Implementing Delaunay triangulations of the Bolza surface

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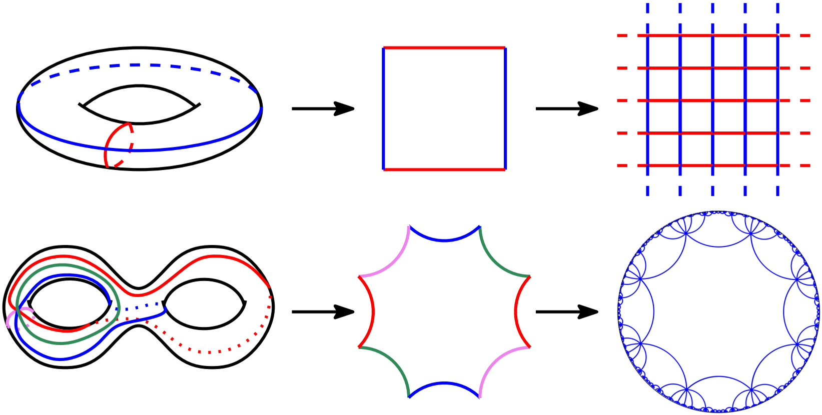
Journées de Géométrie Algorithmique 2017
Aussois, France

Outline

Outline

Motivation

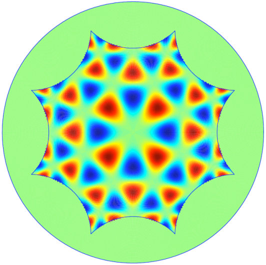
Periodic triangulations: Euclidean vs hyperbolic



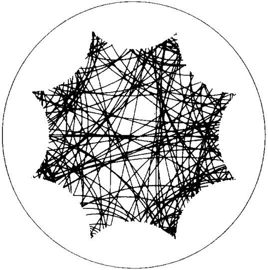
Motivation

Applications

[Sausset, Tarjus, Viot]



[Chossat, Faye, Faugeras]

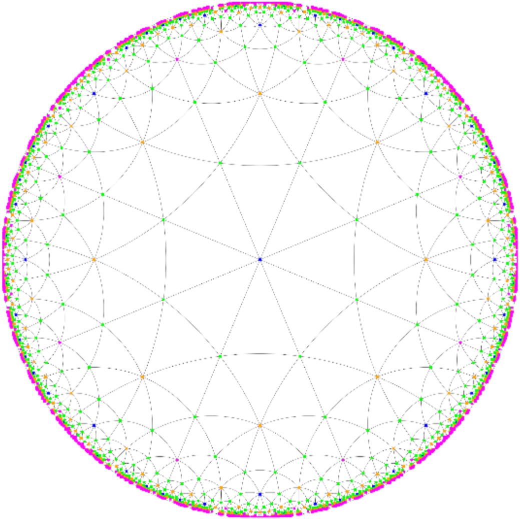


[Balazs, Voros]

Motivation

Beautiful groups

- Fuchsian groups
- finitely presented groups
- triangle groups
- ...



State of the art

Closed Euclidean manifolds

- Algorithms 2D [Mazón, Recio], 3D [Dolbilin, Huson], *d*D [Caroli, Teillaud, DCG'16]
- Software (square/cubic flat torus) 2D [Kruithof], 3D [Caroli, Teillaud]

CGAL

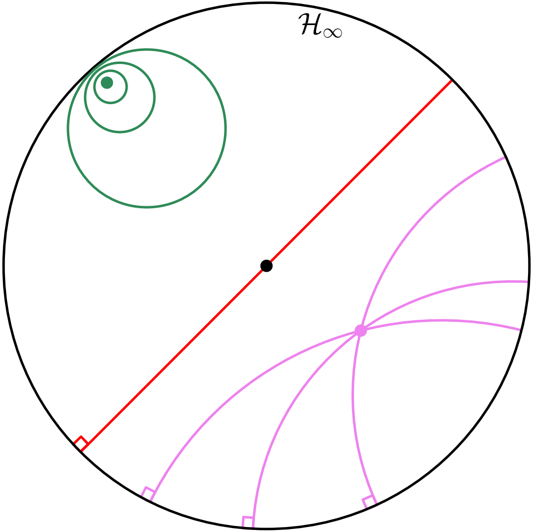
Closed hyperbolic manifolds

- Algorithms 2D, genus 2 [Bogdanov, Teillaud, Vegter, SoCG'16]
- Software (Bolza surface) [*I.*, Teillaud, SoCG'17]

CGAL

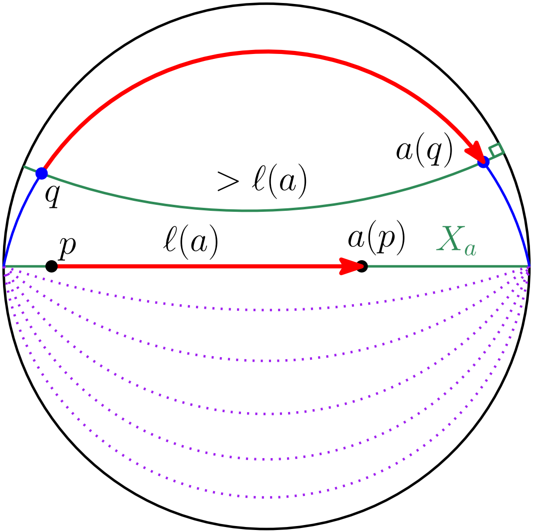
Outline

Poincaré model of the hyperbolic plane $\mathbb{H}^2$



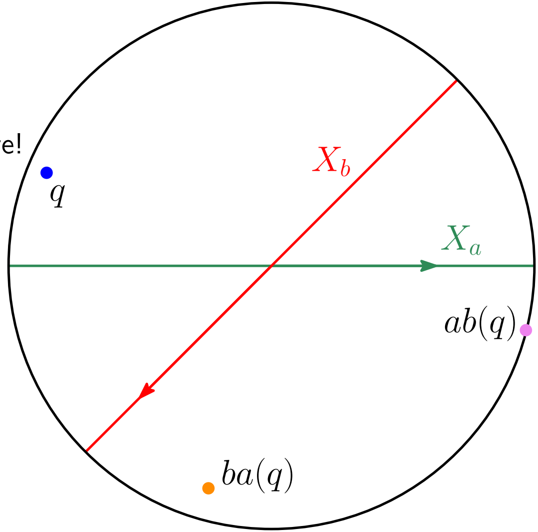
Hyperbolic translations

special case
axis = diameter



Hyperbolic translations

non-commutative!

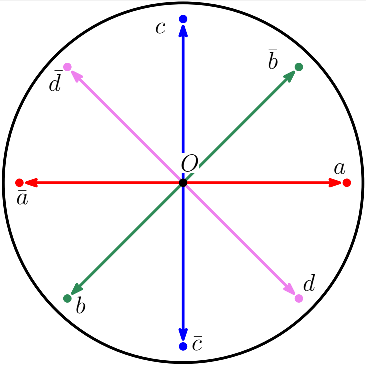


Bolza surface

What is it?

- Closed, compact, orientable surface of genus 2.
- Constant negative curvature $\longrightarrow$ locally hyperbolic metric.
- The most symmetric of all genus-2 surfaces.

Bolza surface



Fuchsian group $\mathcal{G}$ with finite presentation

$$\mathcal{G} = \left\langle a, b, c, d \mid abcd\overline{a}\overline{b}\overline{c}\overline{d} \right\rangle$$

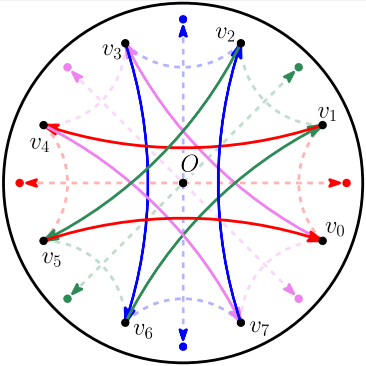
$\mathcal{G}$ contains only translations (and $\mathbb{1}$)

Bolza surface

$$\mathcal{M} = \mathbb{H}^2/\mathcal{G}$$

with projection map $\pi_{\mathcal{M}} : \mathbb{H}^2 \to \mathcal{M}$

Bolza surface



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Bolza surface

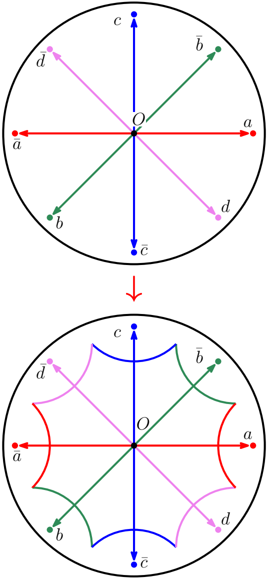
$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

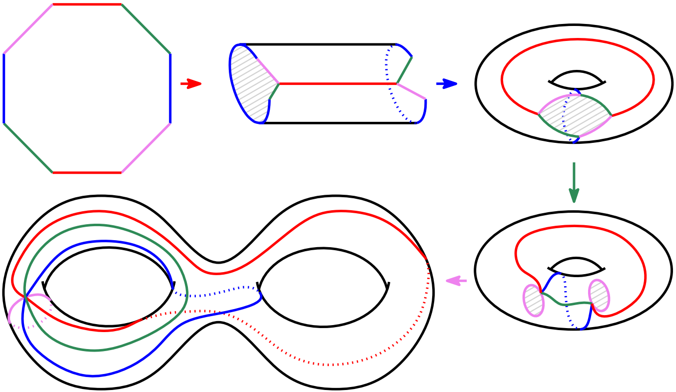
with projection map $\pi_{\mathcal{M}} : \mathbb{H}^2 \to \mathcal{M}$

$$\mathcal{A} = \left[a, \overline{b}, c, \overline{d}, \overline{a}, b, \overline{c}, d\right] = \left[g_0, g_1, \ldots, g_7\right]$$

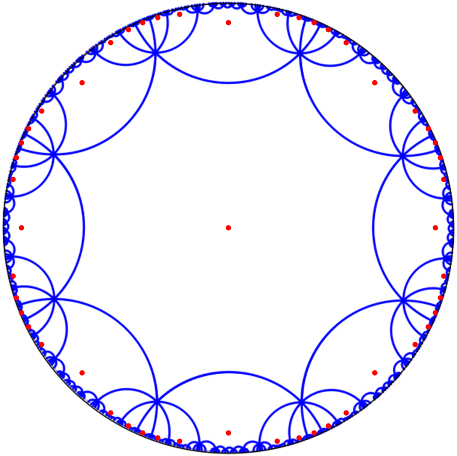
$$g_k = \begin{bmatrix} \alpha & \beta_k \\ \overline{\beta}_k & \overline{\alpha} \end{bmatrix}, \quad g_k(z) = \frac{\alpha z + \beta_k}{\overline{\beta}_k z + \overline{\alpha}}, \quad \alpha = 1 + \sqrt{2}, \quad \beta_k = e^{ik\pi/4}\sqrt{2\alpha}$$

Bolza surface



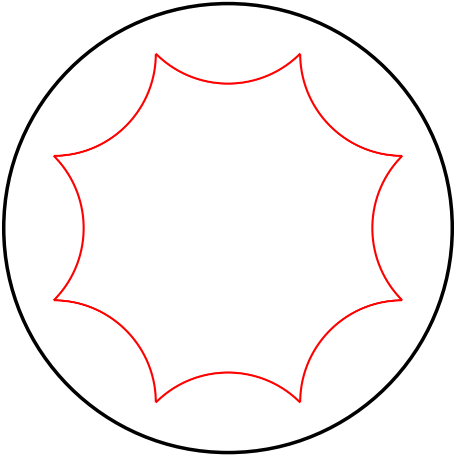


Hyperbolic octagon



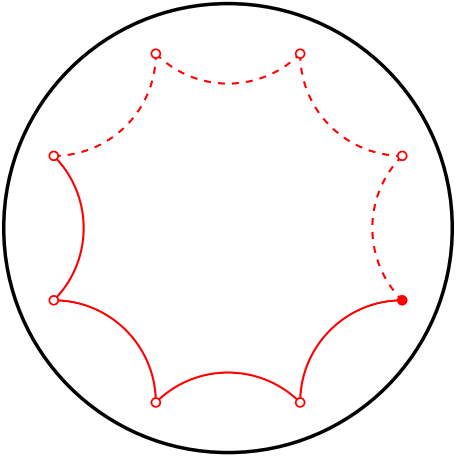
Voronoi diagram of $\mathcal{G}O$

Hyperbolic octagon



Fundamental domain $\mathcal{D}_O$ = Dirichlet region of O

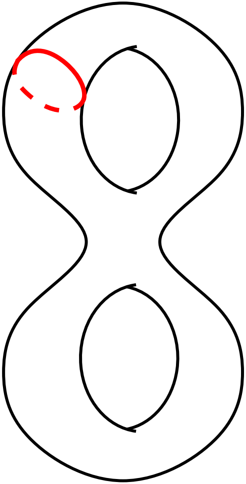
Hyperbolic octagon



"Original" domain $\mathcal{D}$: contains exactly one point of each orbit

Outline

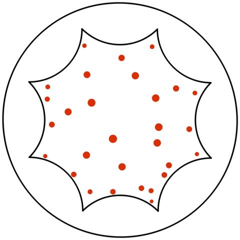
Criterion



Systole $\mathrm{sys}(\mathcal{M})$ = minimum length of a non-contractible loop on $\mathcal{M}$

Criterion

Systole $\mathrm{sys}\,(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$



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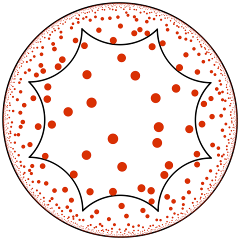

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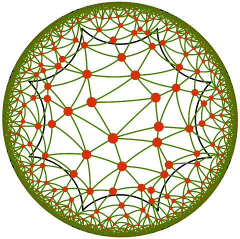

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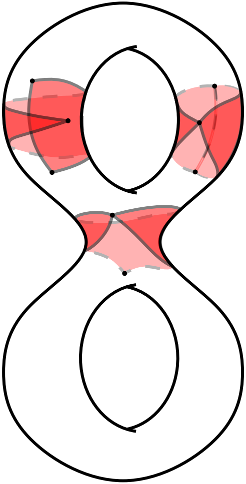
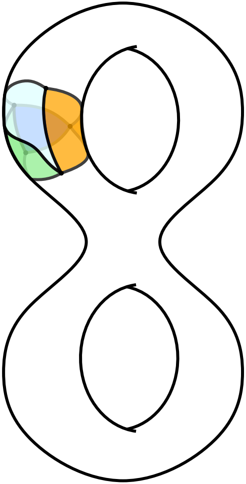


Systole $\mathrm{sys}\,(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

$$\pi_{\mathcal{M}}(\,DT_{\mathbb{H}}\,(\mathcal{G}S)\,)$$

Criterion



Systole $\operatorname{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

S set of points in $\mathbb{H}^2$

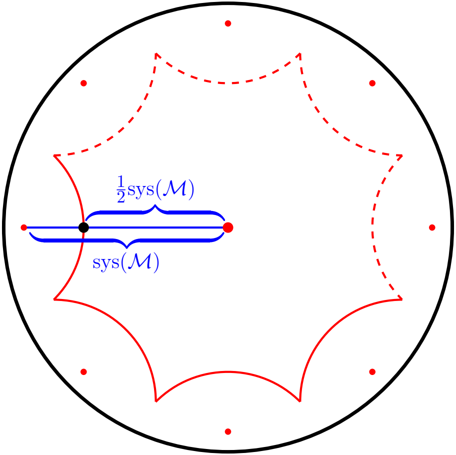
$\delta_S =$ diameter of largest disks in $\mathbb{H}^2$ not containing any point of $\mathcal{G}S$

$\delta_S < \frac{1}{2} \operatorname{sys}(\mathcal{M})$

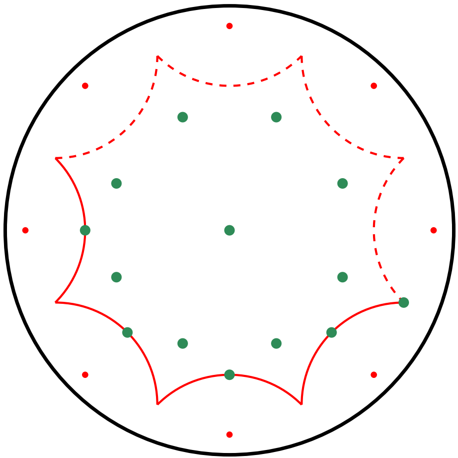
$\Longrightarrow \pi_{\mathcal{M}}(DT_{\mathbb{H}}(\mathcal{G}S)) = DT_{\mathcal{M}}(S)$
is a simplicial complex

$\Longrightarrow$ The usual incremental algorithm can be used [Bowyer]

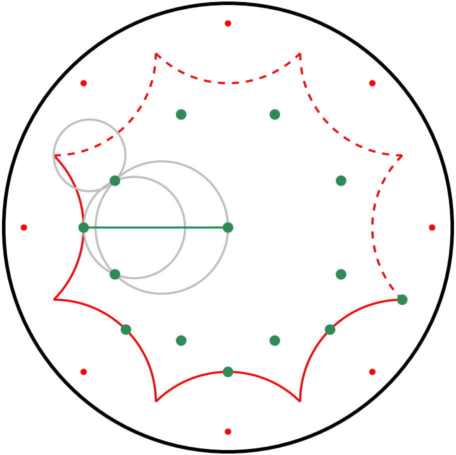
Systole on the octagon



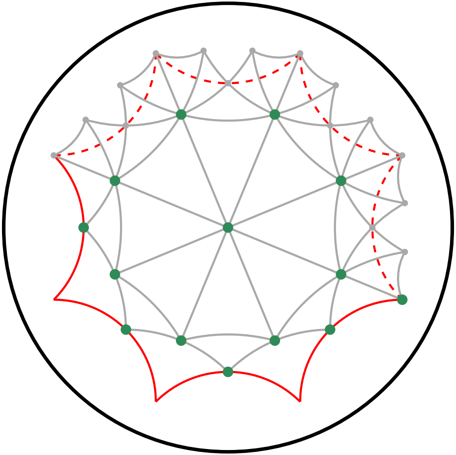
Set of dummy points



Set of dummy points vs. criterion



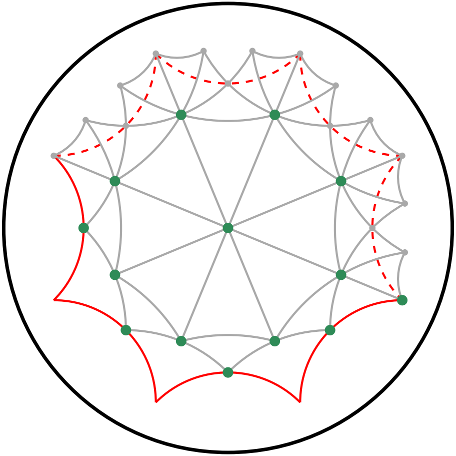
Delaunay triangulation of the dummy points



Delaunay triangulation of the Bolza surface

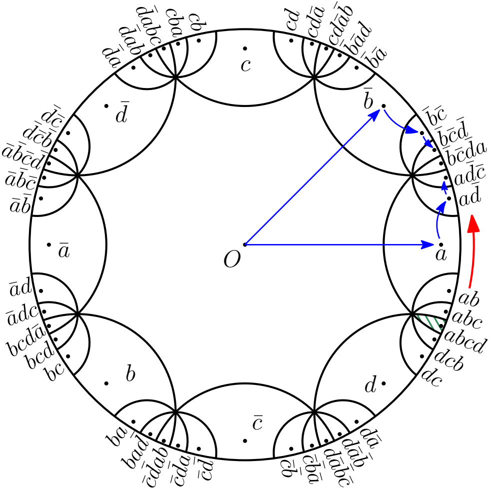
Algorithm:

1. initialize with dummy points
2. insert points in S
3. remove dummy points



Outline

Notation



$g(O),\ g \in \mathcal{G}$, denoted as g

$\mathcal{D}_g = g(\mathcal{D}_O),\ g \in \mathcal{G}$

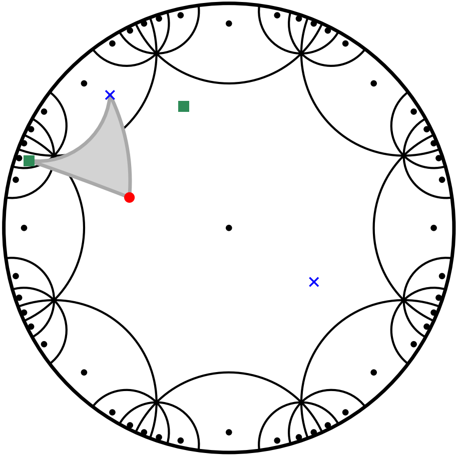
$\mathcal{N} = \{g \in \mathcal{G} \mid \mathcal{D}_g \cap \mathcal{D}_O \neq \emptyset\}$

$$\mathcal{D}_{\mathcal{N}} = \bigcup_{g \in \mathcal{N}} \mathcal{D}_g$$

Property of $DT_{\mathbb{H}}(\mathcal{G}S)$

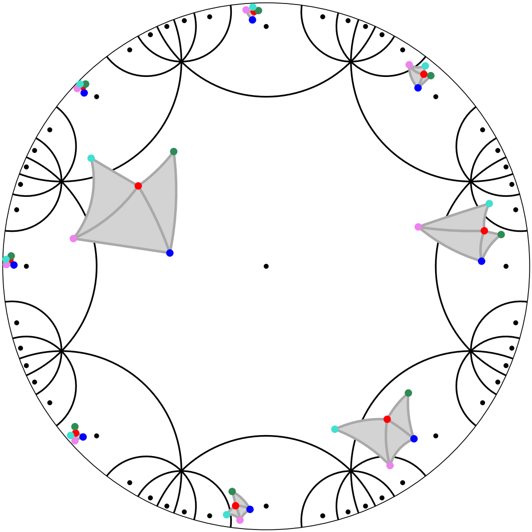
$S \subset \mathcal{D}$ input point set
s.t. criterion $\delta_S < \frac{1}{2}\,\mathrm{sys}(\mathcal{M})$ holds

σ face of $DT_{\mathbb{H}}(\mathcal{G}S)$ with at least one vertex in $\mathcal{D}$
$\longrightarrow$ σ is contained in $\mathcal{D}_{\mathcal{N}}$



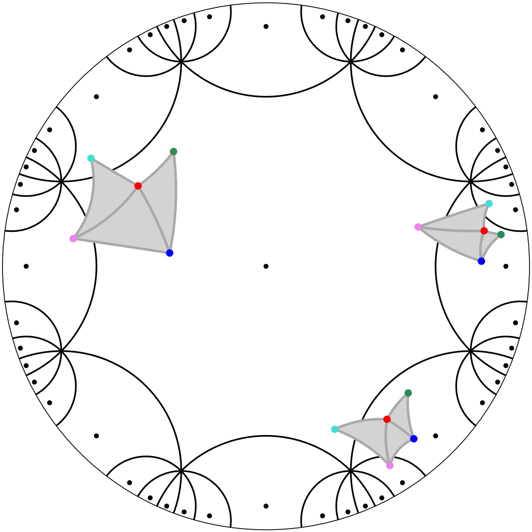
Canonical representative of a face

Each face of $DT_{\mathcal{M}}(S)$ has infinitely many pre-images in $DT_{\mathbb{H}}(\mathcal{G}S)$



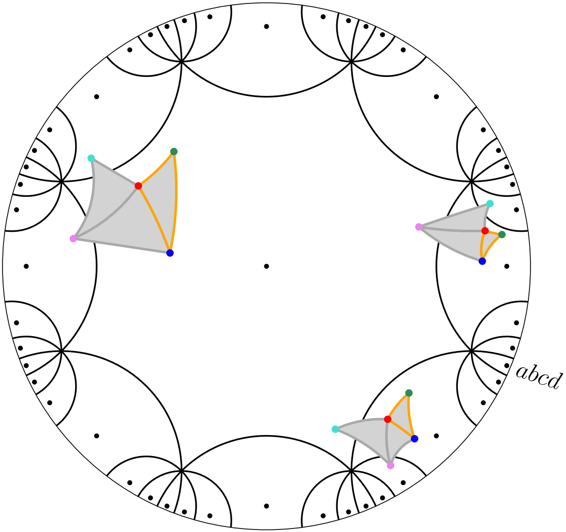
Canonical representative of a face

at least one pre-image with at least one vertex in $\mathcal{D}$



Canonical representative of a face

Case: face with 3 vertices in $\mathcal{D}$



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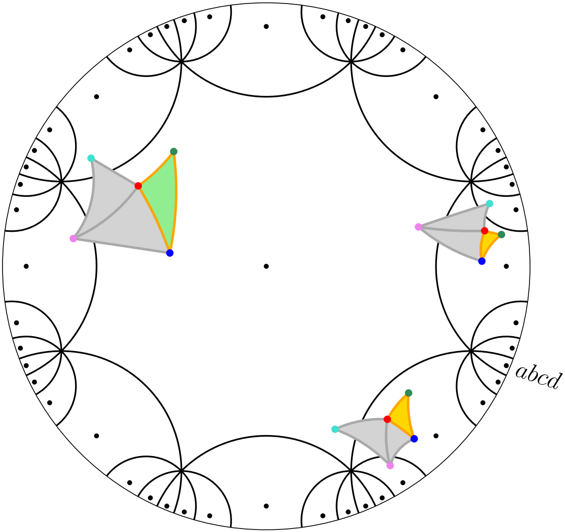

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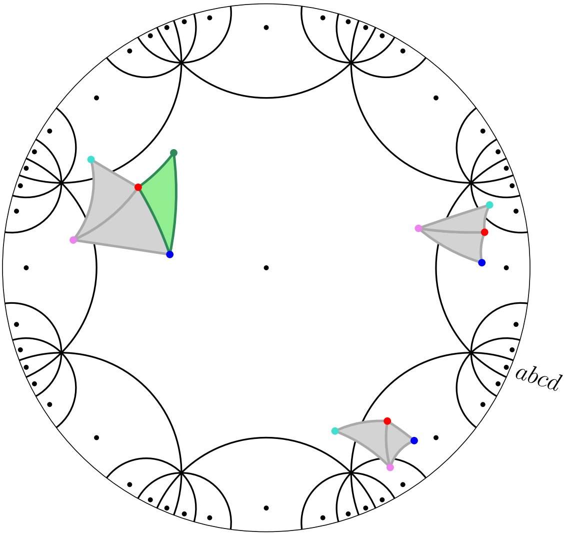
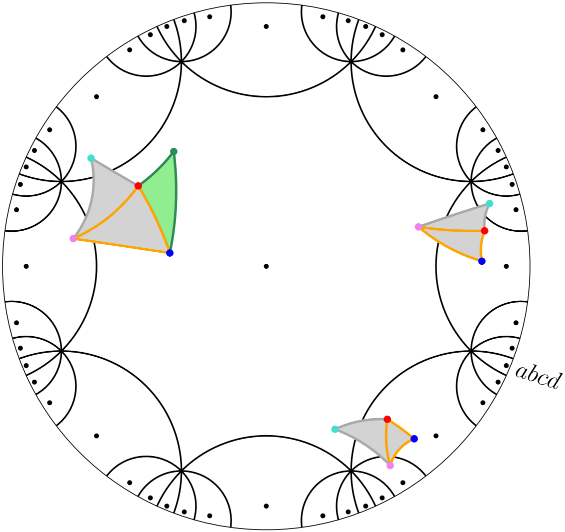


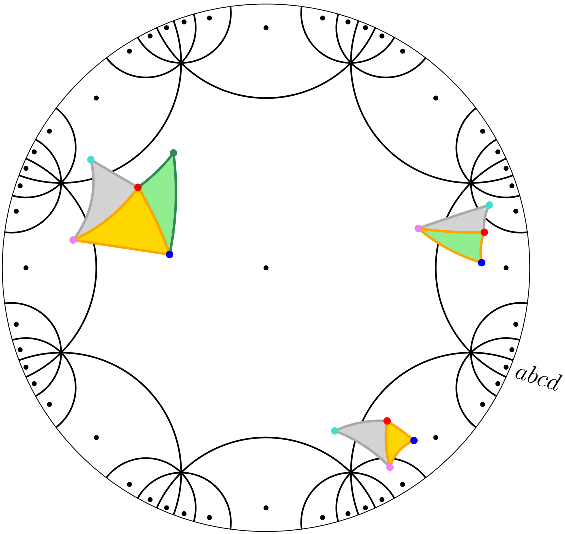
Canonical representative of a face

Case: face with 2 vertices in $\mathcal{D}$



Canonical representative of a face

Case: face with 2 vertices in $\mathcal{D}$



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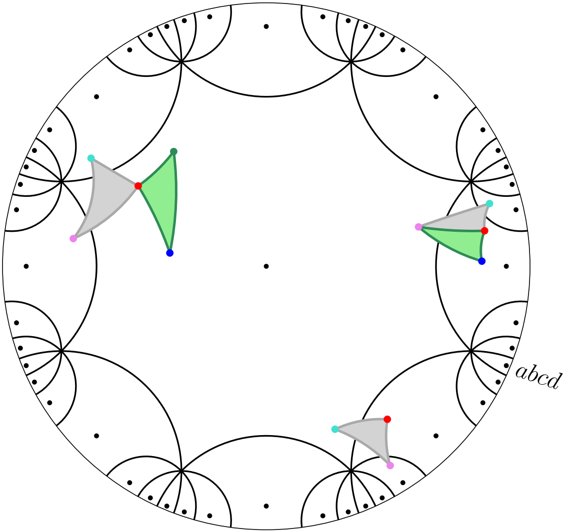
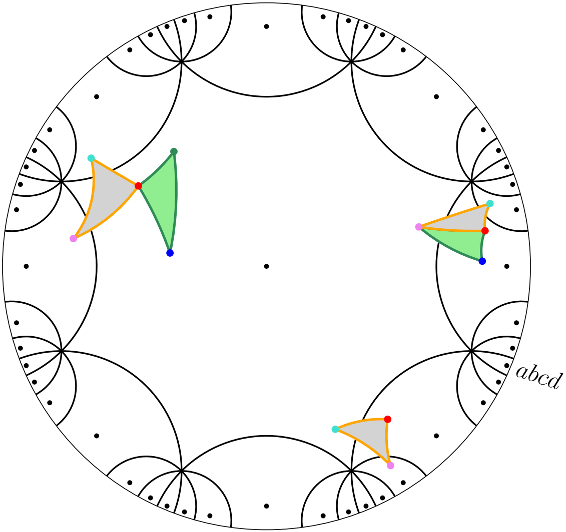


Canonical representative of a face

Case: face with 1 vertex in $\mathcal{D}$



Canonical representative of a face

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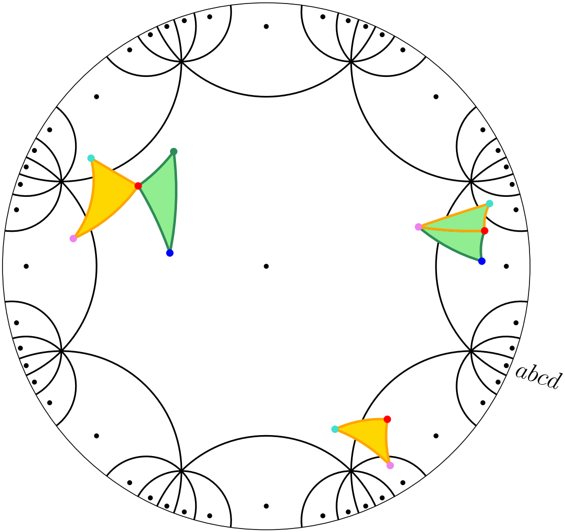
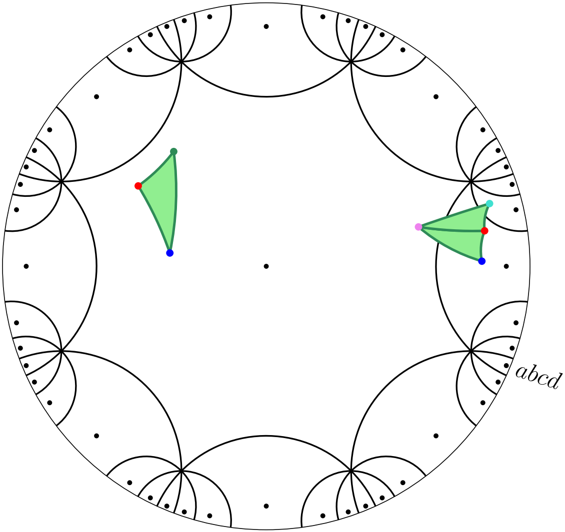
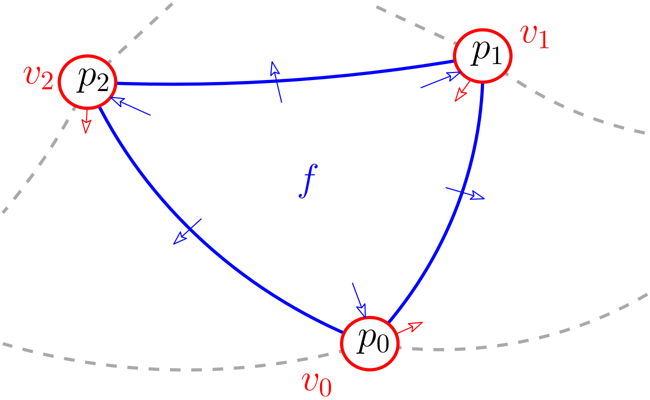


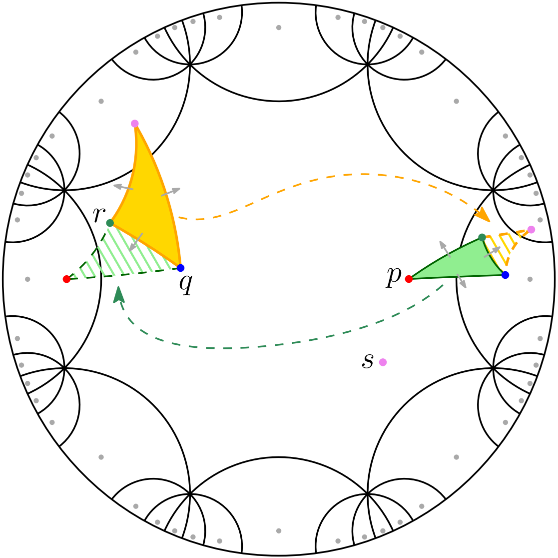
Canonical representative of a face



CGAL Triangulations



Face of $DT_{\mathcal{M}}(S)$



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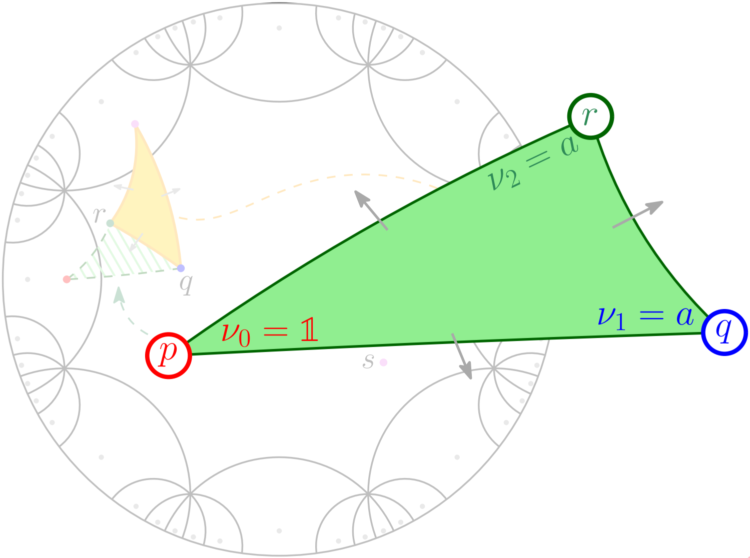
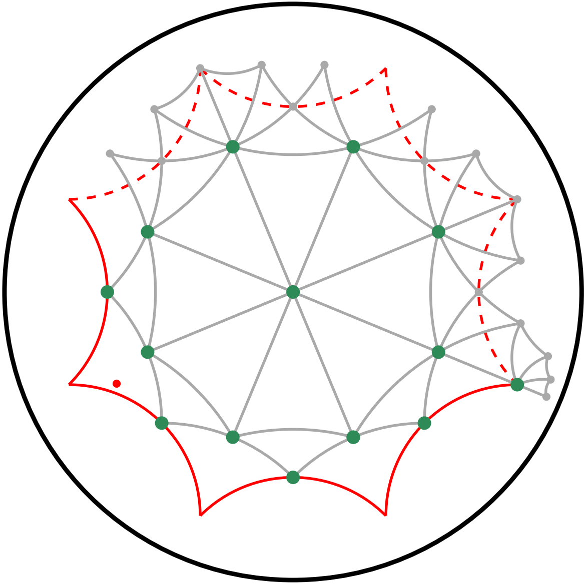


Outline

Point Location



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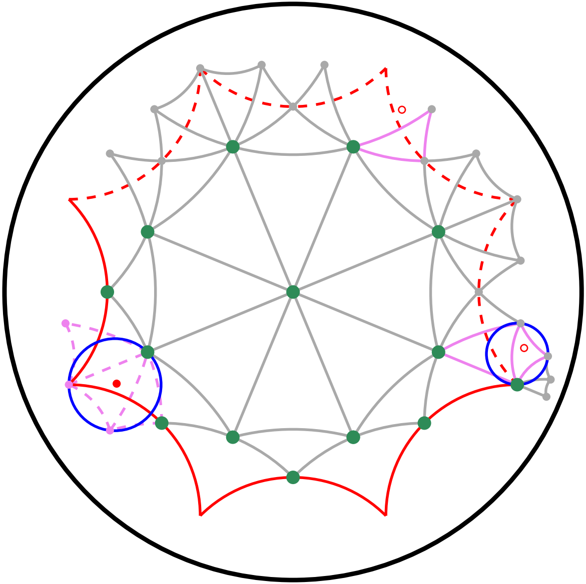

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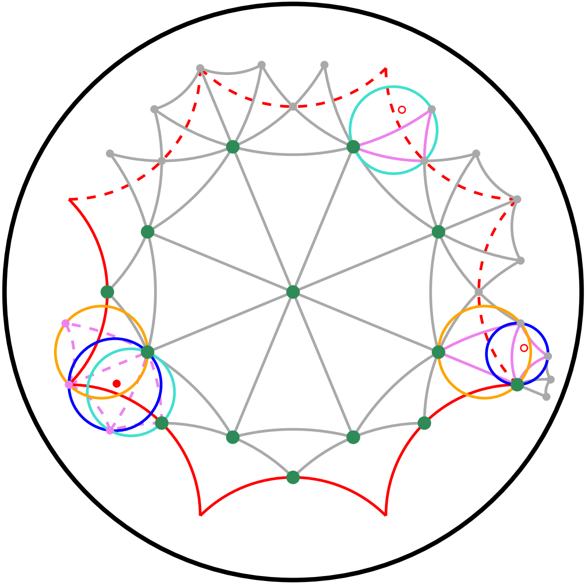

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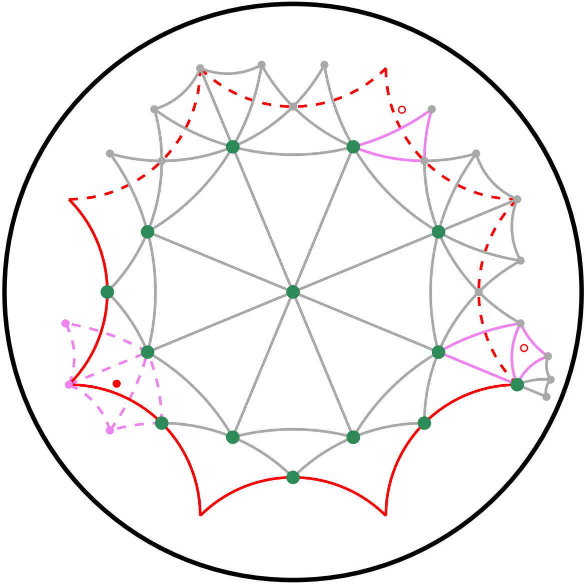
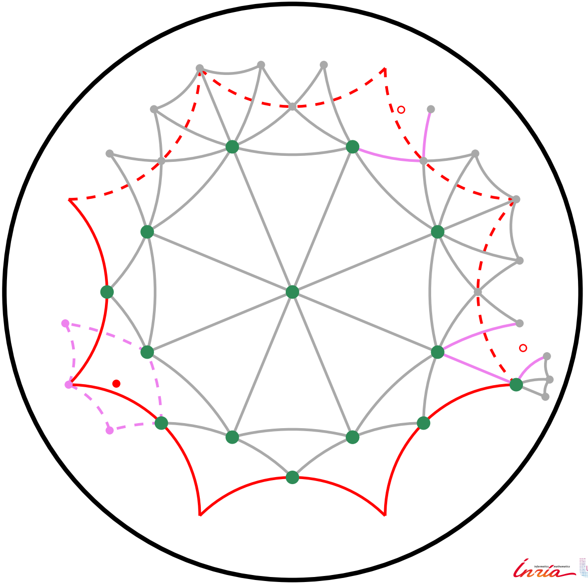


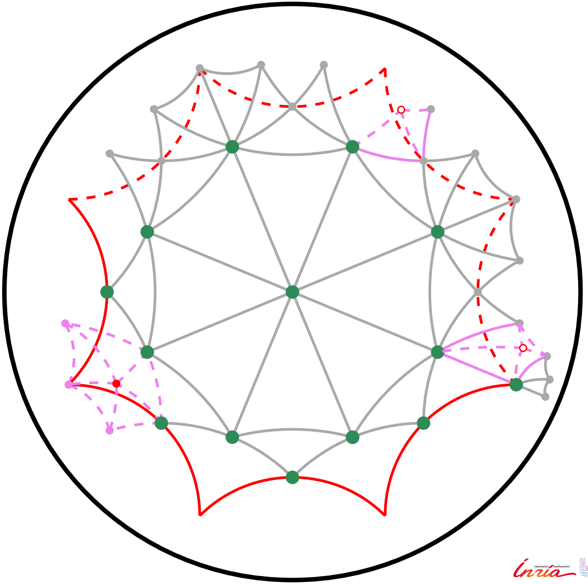
Point Insertion

“hole” = topological disk



Point Insertion

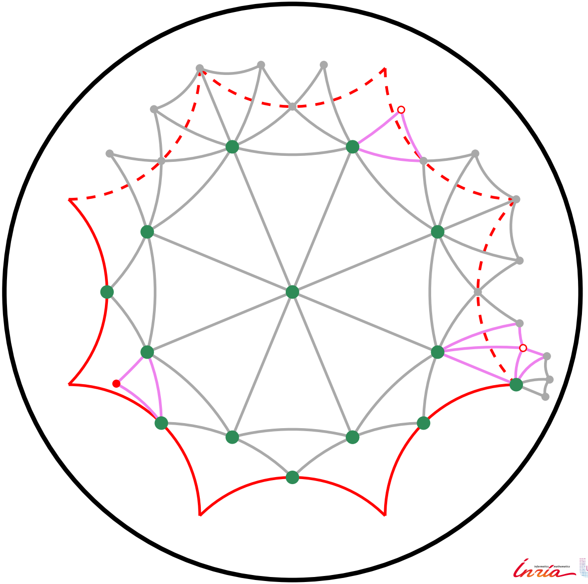
"hole" = topological disk



Point Insertion

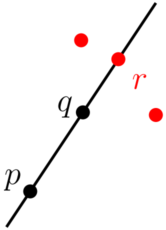
Computations
on translations

Dehn's algorithm
(slightly modified)

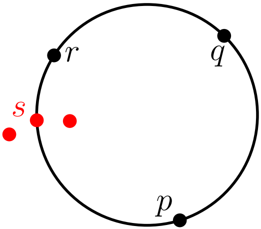


Predicates

$$Orientation\,(p,q,r) = \text{sign}\begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix}$$



$$InCircle\,(p,q,r,s) = \text{sign}\begin{vmatrix} p_x & p_y & p_x^2+p_y^2 & 1 \\ q_x & q_y & q_x^2+q_y^2 & 1 \\ r_x & r_y & r_x^2+r_y^2 & 1 \\ s_x & s_y & s_x^2+s_y^2 & 1 \end{vmatrix}$$



Predicates

Suppose that the points in S are rational.

Input of the predicates can be images of these points under $\nu \in \mathcal{N}$.

$$g_k(z) = \frac{\alpha z + e^{ik\pi/4}\sqrt{2\alpha}}{e^{-ik\pi/4}\sqrt{2\alpha}z + \alpha}, \quad \alpha = 1 + \sqrt{2}, \quad k = 0, 1, ..., 7$$

- the *Orientation* predicate has algebraic degree at most 20
- the *InCircle* predicate has algebraic degree at most 72

Point coordinates represented with `CORE::Expr`
$\longrightarrow$ (filtered) exact evaluation of predicates

Outline

Demo

The code is on GitHub! Let's see a demo.

Experiments

Fully dynamic implementation

1 million random points

- CGAL Euclidean DT (double) — $\sim$ 1 sec.
- CGAL Euclidean DT (CORE::Expr) — $\sim$ 13 sec.
- Hyperbolic periodic DT (CORE::Expr) — $\sim$ 34 sec.

Experiments

Fully dynamic implementation

1 million random points

- CGAL Euclidean DT (double) $\sim$ 1 sec.
- CGAL Euclidean DT (CORE::Expr) $\sim$ 13 sec.
- Hyperbolic periodic DT (CORE::Expr) $\sim$ 34 sec.

Predicates

- 0.76% calls to predicates involving translations in $\mathcal{N}$
- responsible for 36% of total time spent in predicates

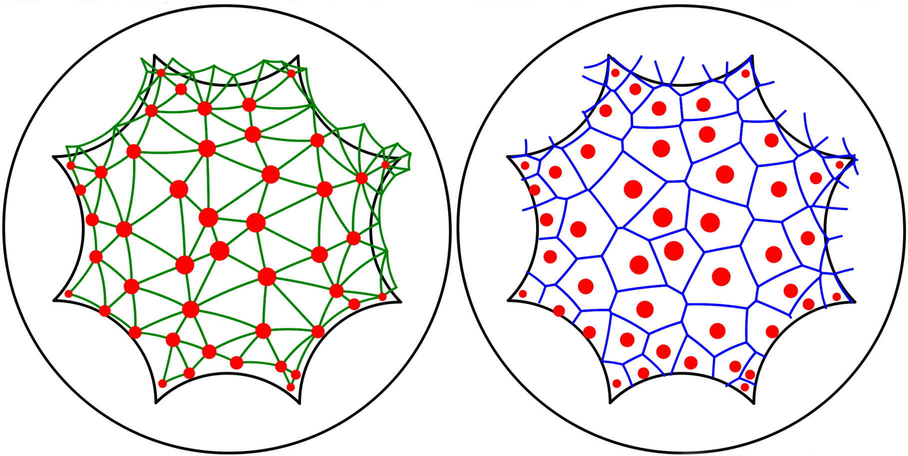
Dummy points can be removed after insertion of 17–72 random points.

Outline

Future work

- Implement 2D periodic hyperbolic mesh
- Algorithm for:
  - More general genus-2 surfaces
  - Surfaces of genus > 2

Thank you for your attention! 🎅



Source code and Maple sheets available online:

`https://members.loria.fr/Monique.Teillaud/DT_Bolza_SoCG17/`