Delaunay triangulations of symmetric hyperbolic surfaces

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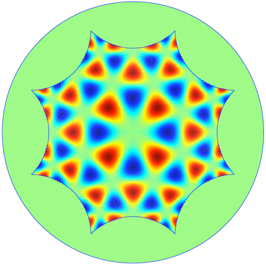
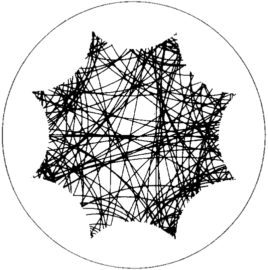
Outline

Motivation

[Sausset, Tarjus, Viot '08]



[Chossat, Faye, Faugeras '11]



[Balazs, Voros '86]

State of the art

Closed Euclidean manifolds

- Algorithms 2D [Mazón, Recio '97], 3D [Dolbilin, Huson '97], *d*D [Caroli, Teillaud '16]
- Software (square/cubic flat torus) 2D [Kruithof '13], 3D [Caroli, Teillaud '09]

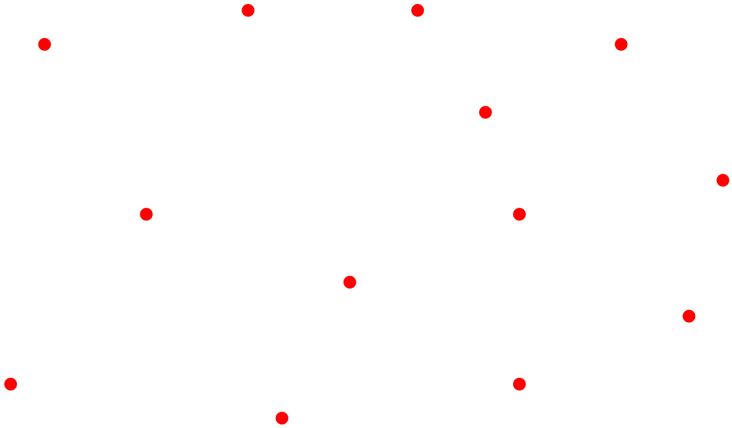
CGAL

Closed hyperbolic manifolds

- Algorithms 2D, genus 2 [Bogdanov, Teillaud, Vegter, SoCG'16]
- Software (Bolza surface) [*I.*, Teillaud, SoCG'17]

→ submitted to CGAL

Delaunay triangulations in the Euclidean plane



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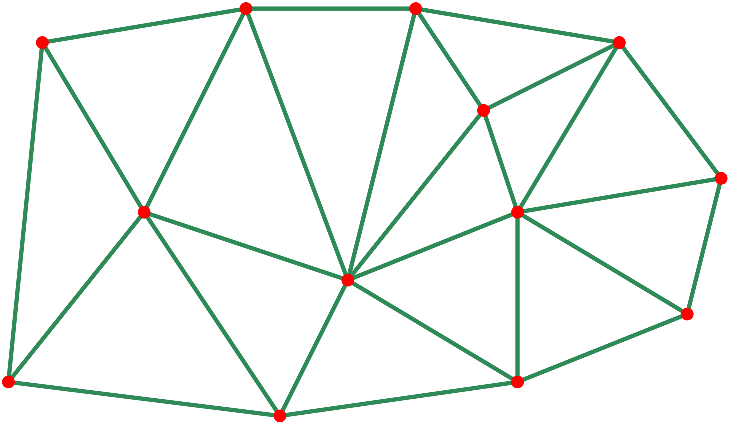
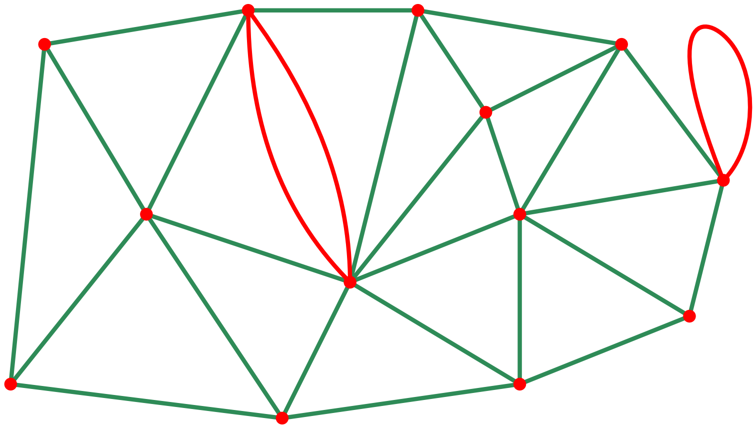


Delaunay triangulations in the Euclidean plane

triangulation = simplicial complex!



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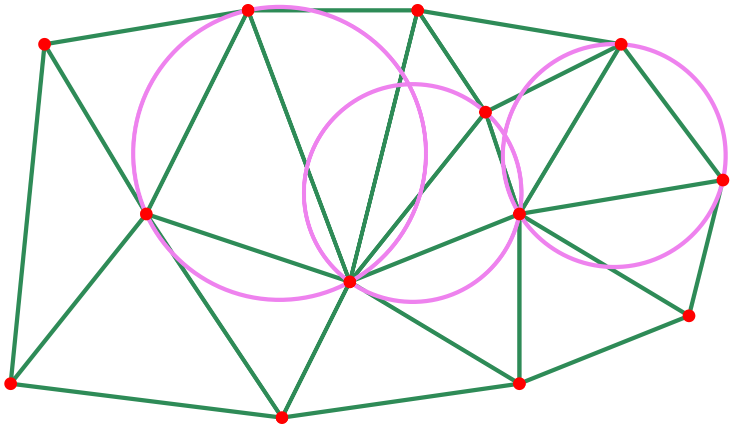
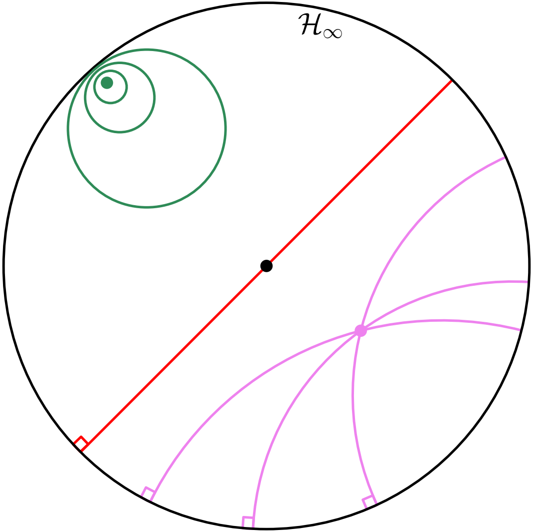


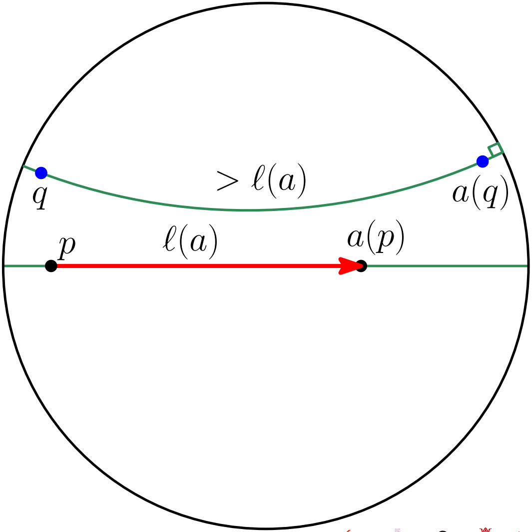
Outline

Poincaré model of the hyperbolic plane $\mathbb{H}^2$



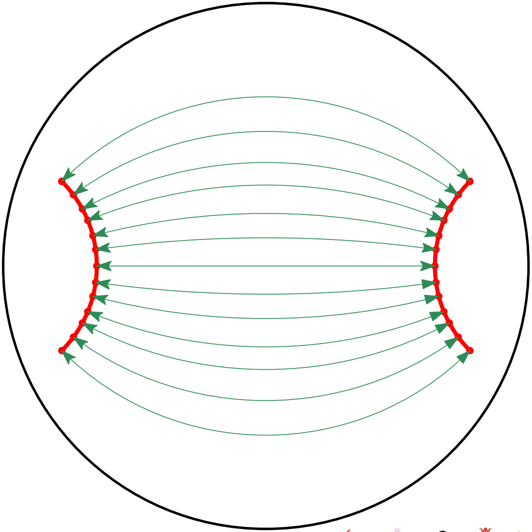
Hyperbolic translations

Special case: axis = diameter



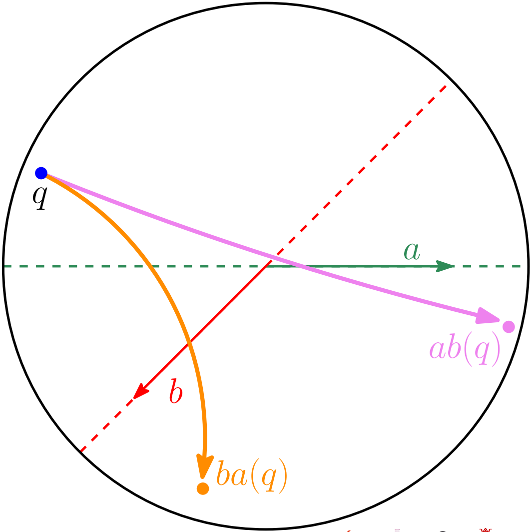
Hyperbolic translations

Side-pairing transformation

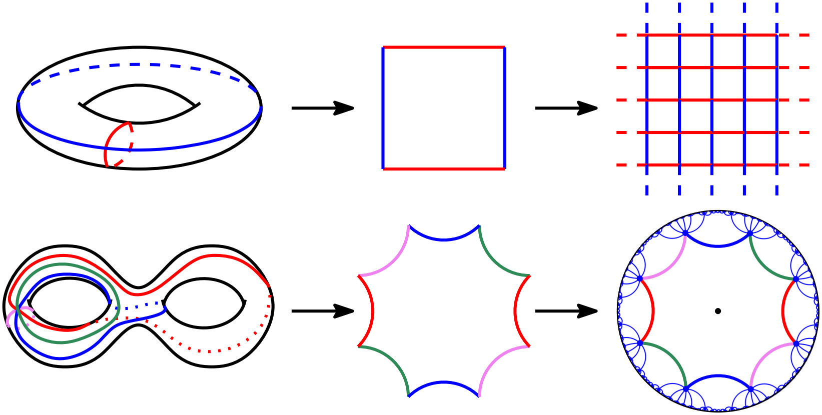


Hyperbolic translations

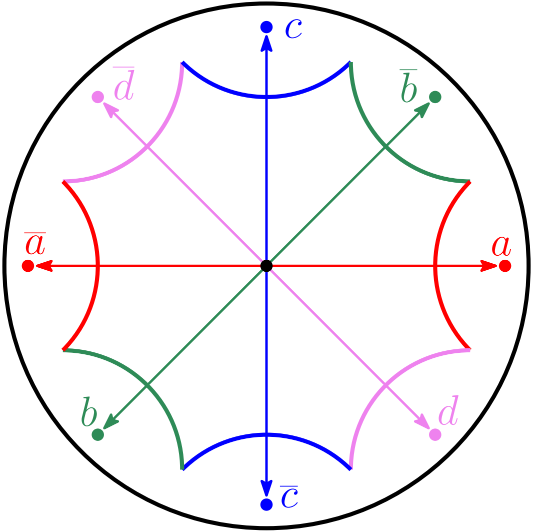
Non-commutative!



Tilings of the Euclidean and hyperbolic planes



Tiling of the hyperbolic plane with octagons



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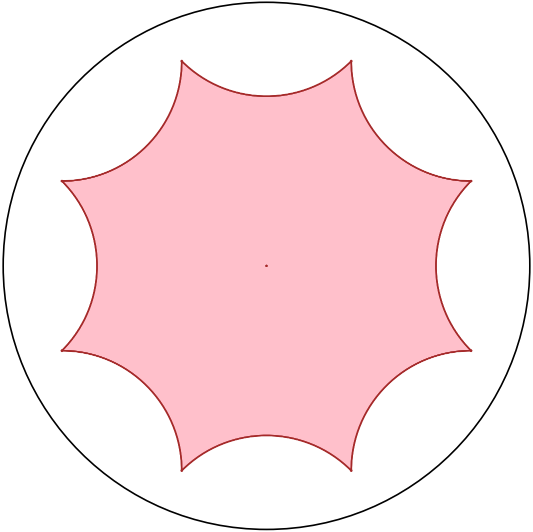

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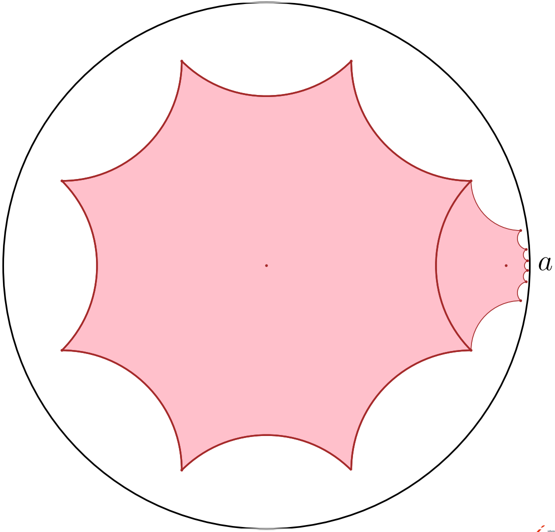

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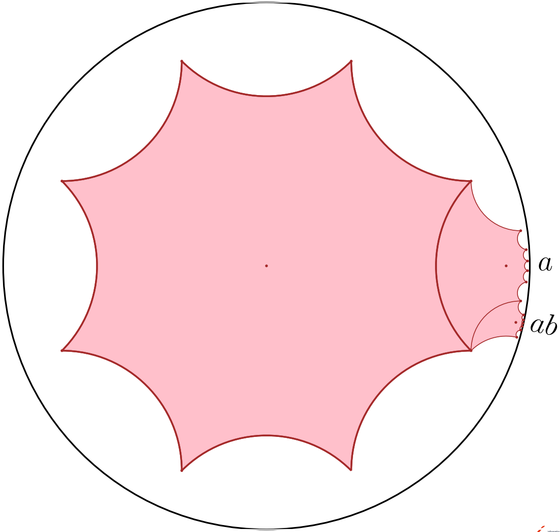

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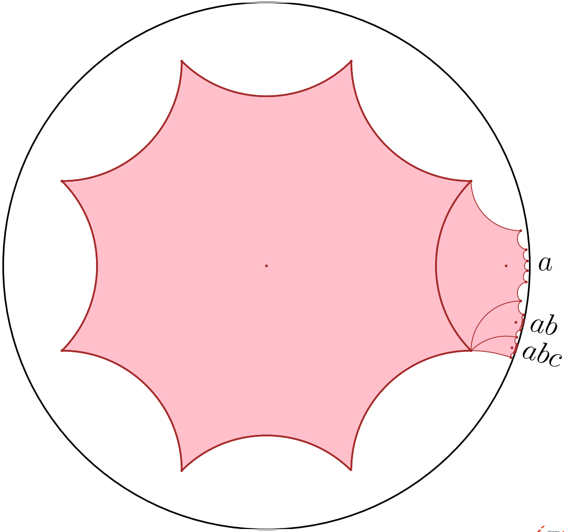

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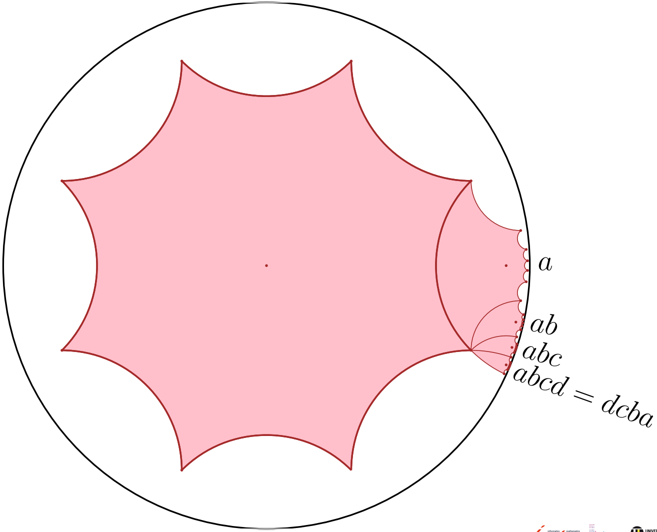

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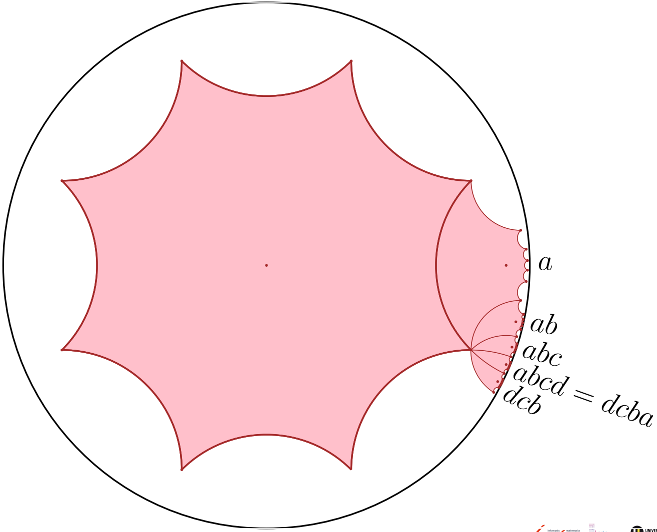

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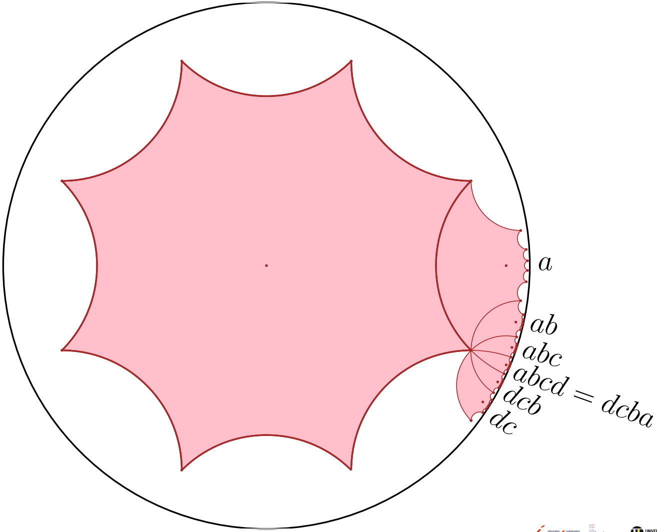

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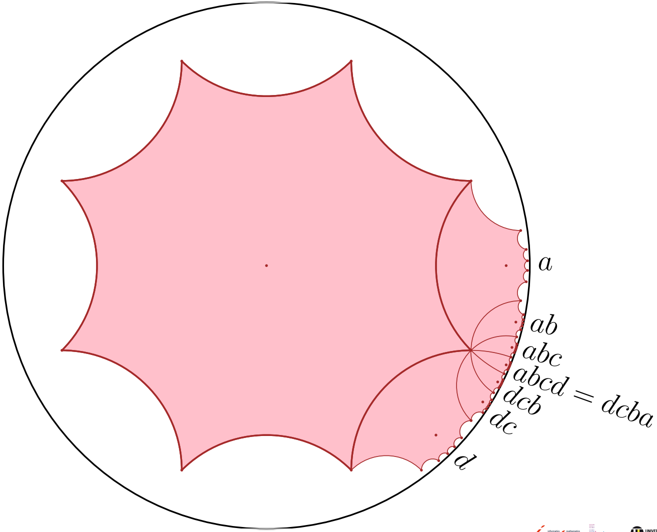

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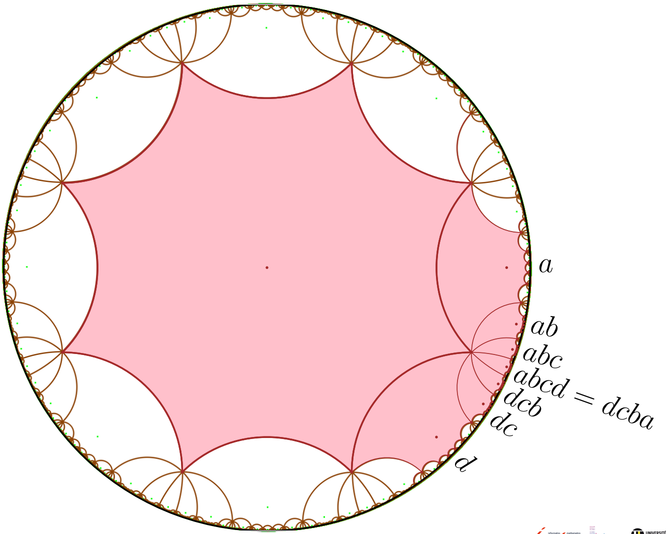

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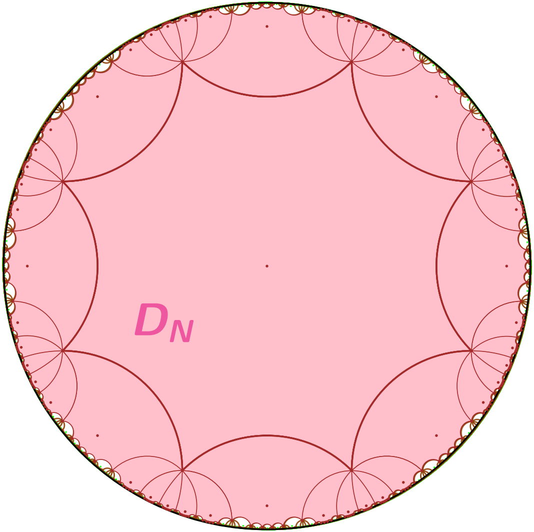
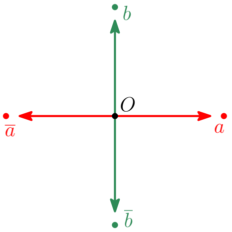


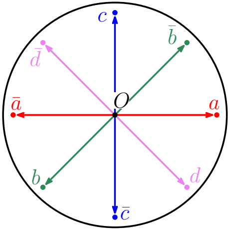
The flat torus and the Bolza surface



Euclidean: translation group

$$\Gamma_1 = \left\langle a, b \mid ab\bar{a}\bar{b} = \mathbb{1} \right\rangle$$

Flat torus: $\mathbb{M}_1 = \mathbb{E}^2/\Gamma_1$
with projection map $\pi_1 : \mathbb{E}^2 \to \mathbb{M}_1$



Hyperbolic: Fuchsian group

$$\Gamma_2 = \left\langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} = \mathbb{1} \right\rangle$$

Bolza surface: $\mathbb{M}_2 = \mathbb{H}^2/\Gamma_2$
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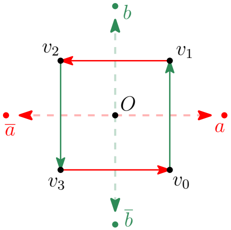

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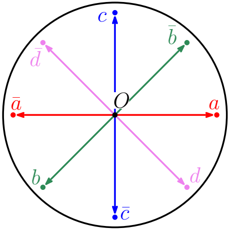

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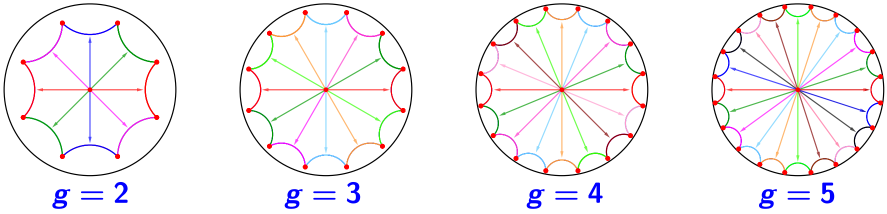
Symmetric hyperbolic surfaces of genus $g \geq 2$



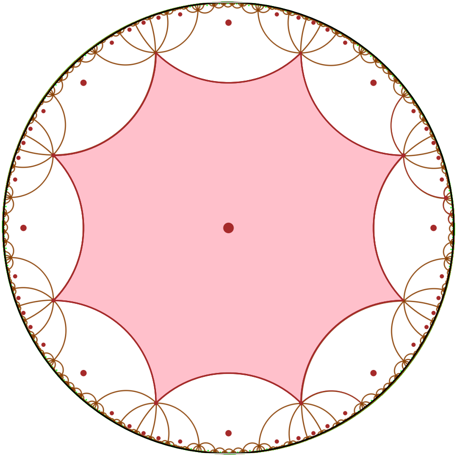
angle sum $= 2\pi$ for all $4g$-gons!

Let Γ_g: Fuchsian group with finite presentation similar to Bolza
$\rightarrow 2g$ generators, single relation

Symmetric hyperbolic surface: $\mathbb{M}_g = \mathbb{H}^2/\Gamma_g, \qquad g \geq 2$

with natural projection mapping $\pi_g : \mathbb{H}^2 \rightarrow \mathbb{M}_g$

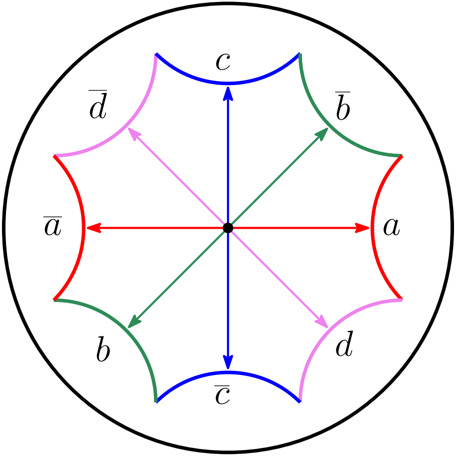
Dirichlet regions



Voronoi diagram of $\Gamma_g O$ for $g = 2$

Dirichlet regions

angle sum $= 2\pi$

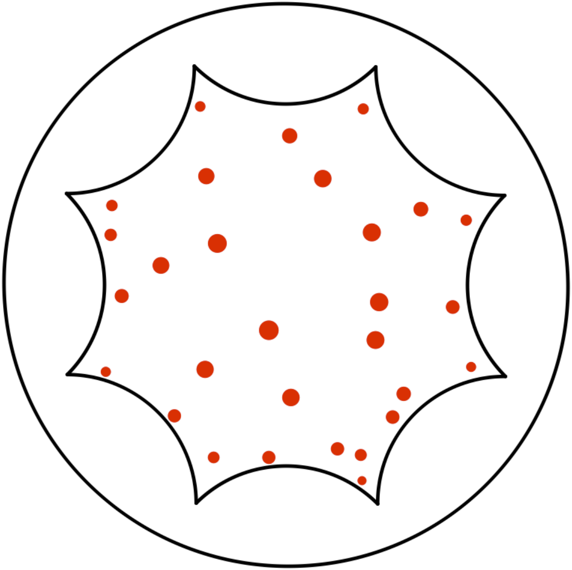


Fundamental domain D_g = Dirichlet region of O for Γ_g

here for $g = 2$

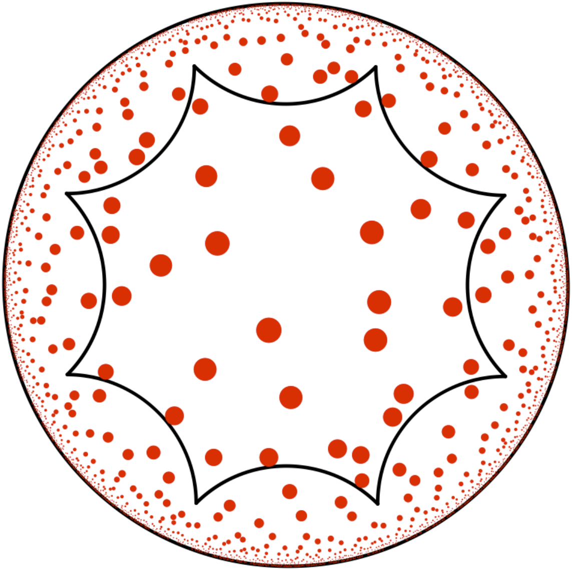
Outline

Validity condition [BTV16]



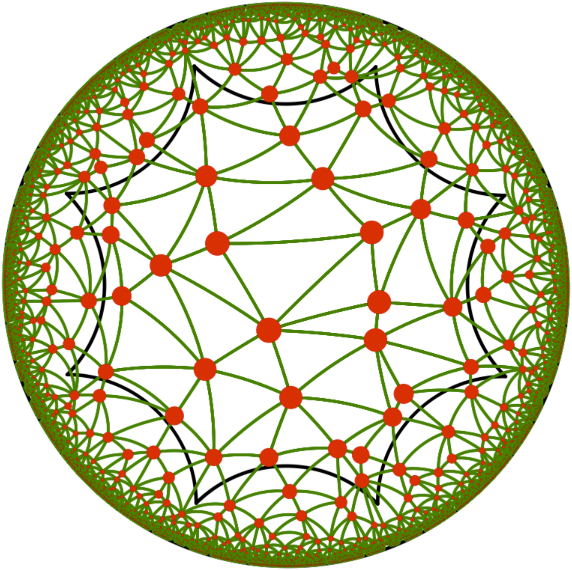
S set of points in D_g

Validity condition [BTV16]



orbits $\Gamma_g S$ in $\mathbb{H}^2$

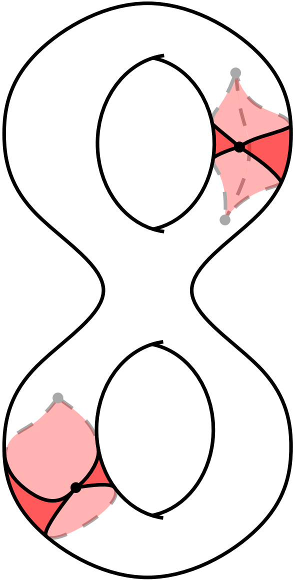
Validity condition [BTV16]



Delaunay triangulation in $\mathbb{H}^2$

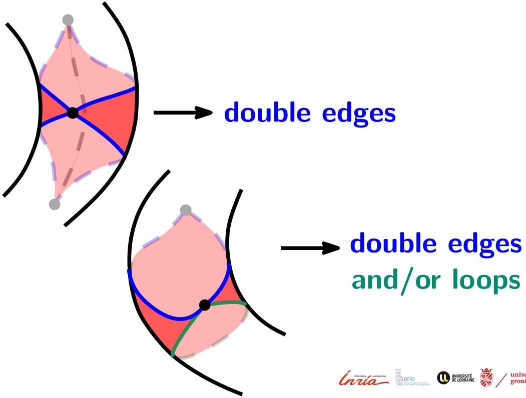
$DT_{\mathbb{H}}(\Gamma_g S)$

Validity condition [BTV16]



projection of $DT_{\mathbb{H}}(\Gamma_g S)$ on the surface $\mathbb{M}_g$

→ not necessarily a simplicial complex!



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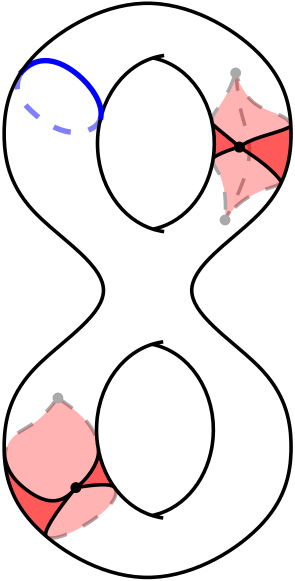
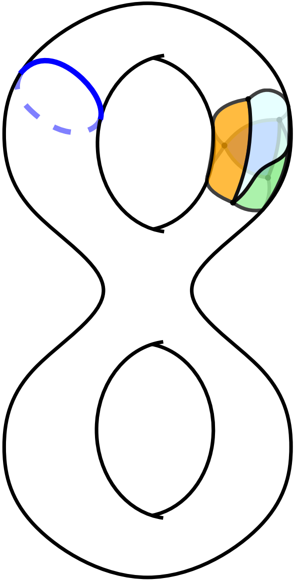


projection of $DT_{\mathbb{H}}(\Gamma_g S)$ on the surface $\mathbb{M}_g$

$\rightarrow$ not necessarily a simplicial complex!

Systole of a surface = minimum length of a non-contractible loop on the surface

Validity condition [BTV16]



projection of $DT_{\mathbb{H}}(\Gamma_g S)$ on the surface $\mathbb{M}_g$

$\rightarrow$ **is a simplicial complex, if**

$$\delta_S < \frac{1}{2}\text{sys}(\mathbb{M}_g), \quad \text{where}$$

$\delta_S =$ diameter of largest disks in $\mathbb{H}^2$ not containing any point of $\Gamma_g S$

$$\boldsymbol{DT_{\mathbb{M}_g}(S) := \pi_g(DT_{\mathbb{H}}(\Gamma_g S))}$$

Computing Delaunay triangulations of $\mathbb{M}_g$

Use set of *dummy points Q_g* that satisfies the validity condition:

$$S := Q_g \bigcup P \Longrightarrow \delta_S < \frac{1}{2}\mathrm{sys}(\mathbb{M}_g) \quad \text{always}$$

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Algorithm for computing Delaunay triangulations of $\mathbb{M}_g$ [BTV16]

- initialize $DT_{\mathbb{M}_g}$ with a set Q_g that satisfies the validity condition;
- insert input points P in the triangulation;
- remove points of Q_g from the triangulation, if possible.

→ condition preserved with insertion of new points
  → diameter of largest empty disks cannot grow

→ final triangulation might contain dummy points
  → if input points too few and/or badly distributed

Outline

Problem statement

To compute $DT_{\mathbb{M}_g}(S)$, we need to *choose* **what** to store.

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Requirement: all input points lie in D_g
$\rightarrow$ unique representative in $D_g \subset \mathbb{H}^2$ for each point on $\mathbb{M}_g$

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$\rightarrow$ unique representative in $D_g \subset \mathbb{H}^2$ for each point on $\mathbb{M}_g$

Question: How to choose a unique representative for each *face*?

Inclusion property

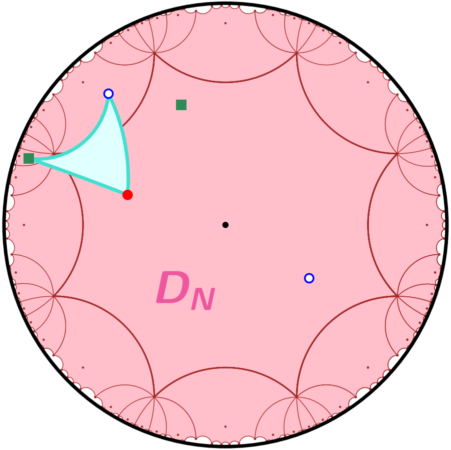
Let $S \subset D_g$ be a point set such that
$$\delta_S < \frac{1}{2}\,\text{sys}(\mathbb{M}_g).$$

Let σ be a face of $DT_{\mathbb{H}}(\Gamma_g S)$ with at least one vertex in D_g
$\Rightarrow$ σ is contained in D_N

Proof:

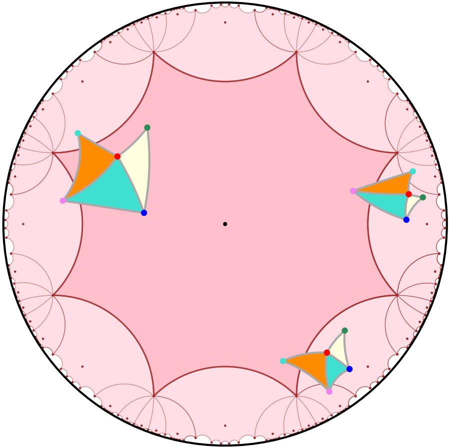
- for $g = 2$ $\rightarrow$ [IT17]
- for $g \geq 2$ $\rightarrow$ [Ebbens 2018]

Matthijs' talk ⇝



Canonical representatives of faces

Canonical representative: face with at least one vertex in D_g
$\rightarrow$ other vertices will be in D_N



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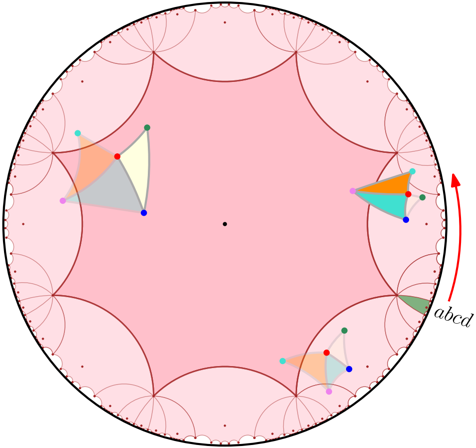
$\rightarrow$ choose the face closest to the "first" Dirichlet neighbor



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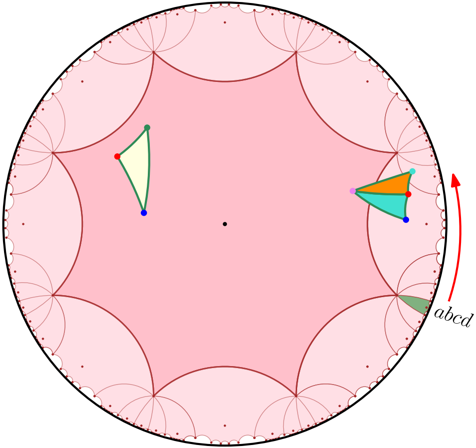
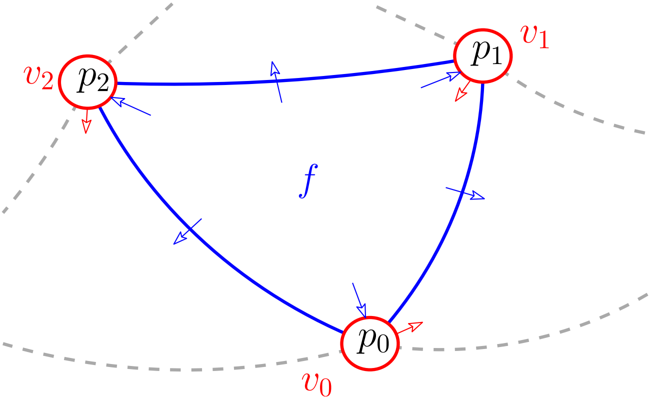
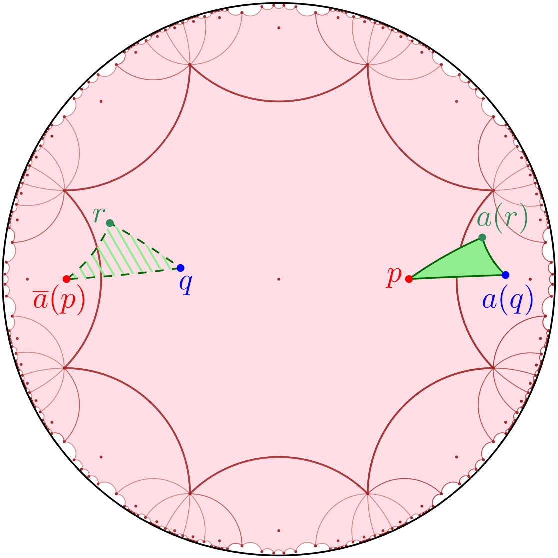


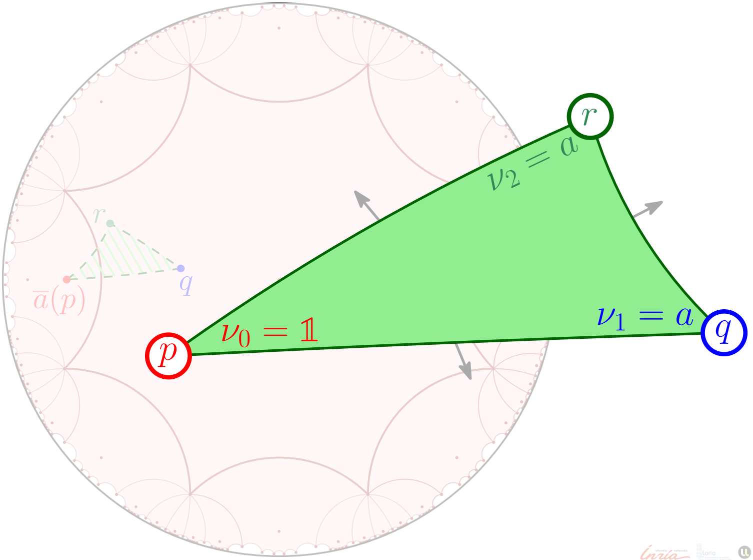
CGAL triangulation data structure



Canonical representatives can cross the boundary



CGAL extended triangulation data structure



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An initial set of dummy points

For $\mathbb{M}_2$, a set of dummy points was given [BTV16]. **In general?**

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The idea is to **generate** dummy points:

1. Start with the set W_g of *Weierstrass points* for $\mathbb{M}_g$
   $\rightarrow$ origin, one vertex, and midpoints of half the sides of the $4g$-gon
2. Compute the images of these points in D_N
3. Compute their hyperbolic Delaunay triangulation in $\mathbb{H}^2$
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[Ebbens, 2018]

$\mathrm{sys}(\mathbb{M}_g) = 2\,\mathrm{arcosh}\left(1 + 2\cos\left(\frac{\pi}{2g}\right)\right)$ Matthijs' talk $\rightsquigarrow$

Outline

Implementation

Available code:

- triangulations in $\mathbb{H}^2$ (non-periodic)
- triangulations of $\mathbb{M}_2$ (periodic)

} https://imiordanov.github.io/code

- generate dummy points with different strategies

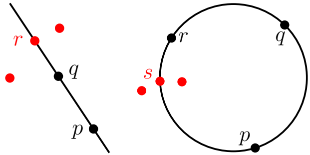
Todo:

- Put the pieces together

Difficulties:

- Numerical operations with `CORE::Expr`

Computer algebraic issues



Combinatorial validity of $DT_{\mathbb{M}_g}$

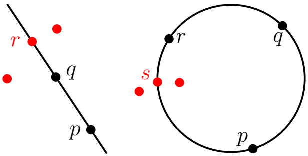
$\Updownarrow$

Exact evaluation of *predicates*

} `CORE::Expr`

Assume rational input points, approximate W-points and circumcenters.

Computer algebraic issues



Combinatorial validity of $DT_{\mathbb{M}_g}$
$\Updownarrow$
Exact evaluation of *predicates*
} `CORE::Expr`

Assume rational input points, approximate W-points and circumcenters.

Hyperbolic translations include *algebraic numbers:*

$$T_k = \begin{bmatrix} \cot(\frac{\pi}{4g}) & \exp(\frac{ik\pi}{2g})\sqrt{\cot^2(\frac{\pi}{4g}) - 1} \\ \exp(-\frac{ik\pi}{2g})\sqrt{\cot^2(\frac{\pi}{4g}) - 1} & \cot(\frac{\pi}{4g}) \end{bmatrix}$$

$\rightarrow$ images of rational points have algebraic coordinates!

Outline

Future directions

- Generalization to arbitrary hyperbolic structures

Thank you!