2D hyperbolic and periodic hyperbolic triangulations

Iordan Iordanov Monique Teillaud

CGAL Developer Meeting, March 2018
Nancy, France

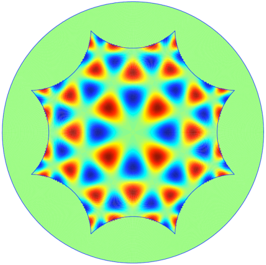
Outline

Outline

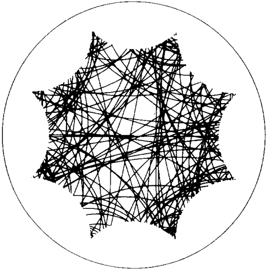
Motivation

Applications

[Sausset, Tarjus, Viot]



[Chossat, Faye, Faugeras]



[Balazs, Voros]

State of the art

Hyperbolic Delaunay simplicial complexes

- Algorithm [Bogdanov, Devillers, Teillaud, JoCG'14]
- Software [Bogdanov, Teillaud, cgal-public-dev]

Closed Euclidean manifolds

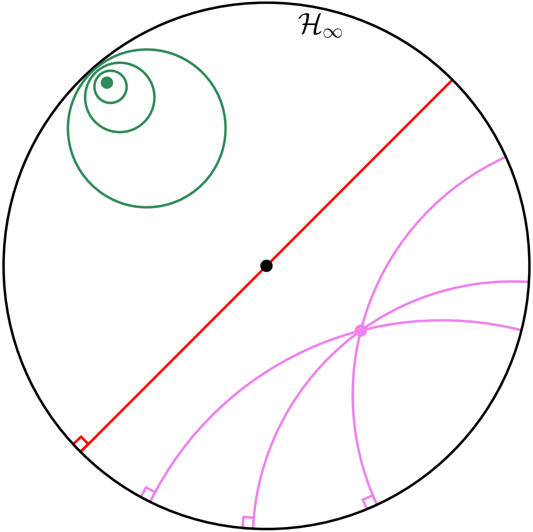
- Algorithms 2D [Mazón, Recio], 3D [Dolbilin, Huson], *d*D [Caroli, Teillaud, DCG'16]
- Software (square/cubic flat torus) CGAL
  - 2D → `Periodic_2_triangulation_2` [Kruithof]
  - 3D → `Periodic_3_triangulation_3` [Caroli, Teillaud]

Closed hyperbolic manifolds

- Algorithms 2D, genus 2 [Bogdanov, Teillaud, Vegter, SoCG'16]
- Software (Bolza surface) [*I.*, Teillaud, SoCG'17]

Outline

Poincaré model of the hyperbolic plane $\mathbb{H}^2$



Hyperbolic Delaunay triangulations

- Hyperbolic circles $\equiv$ Euclidean circles $\Rightarrow$ reuse predicates
- If circumcircle intersects Poincaré disk, simplex is not hyperbolic!
  $\rightarrow$ filter out non-hyperbolic faces and edges
- Only circumcenters (Voronoi points) have non-rational coordinates

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- Demo

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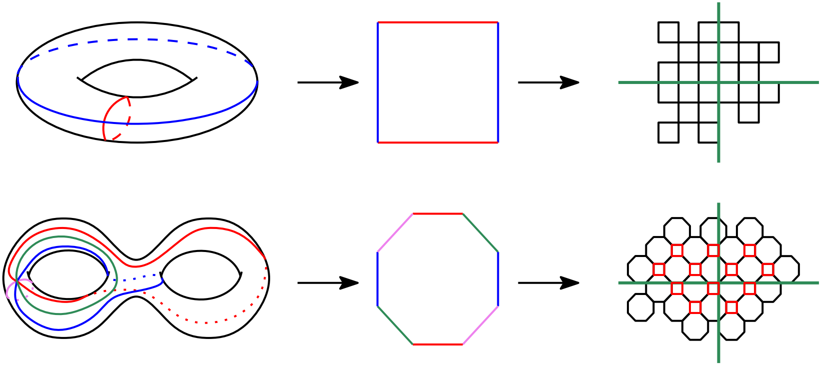
Periodic triangulations in the Euclidean plane



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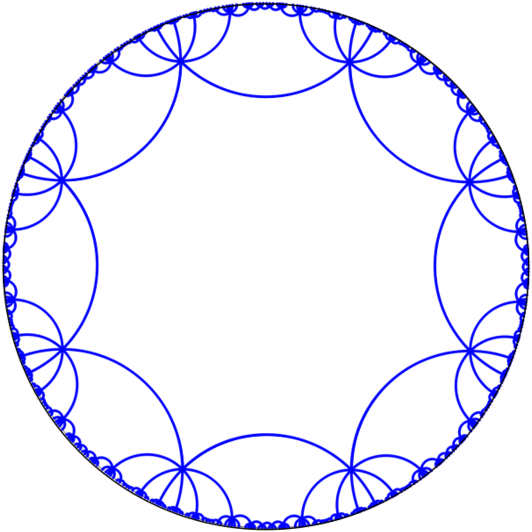
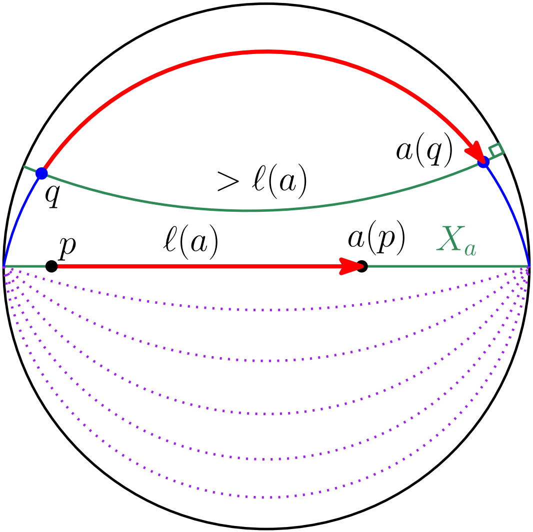
Periodic triangulations in the hyperbolic plane



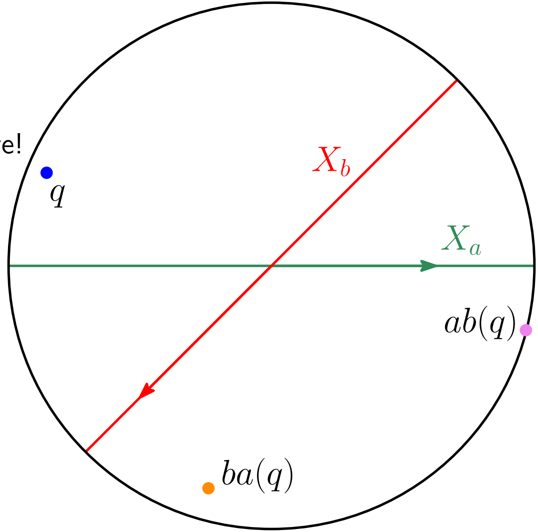
Hyperbolic translations

special case
axis = diameter

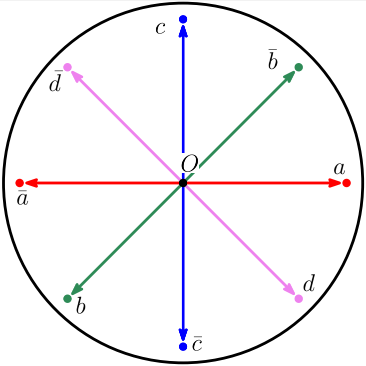


Hyperbolic translations

non-commutative!



Bolza surface



Fuchsian group $\mathcal{G}$ with finite presentation

$$\mathcal{G} = \left\langle a, b, c, d \mid abcd\overline{a}\overline{b}\overline{c}\overline{d} \right\rangle$$

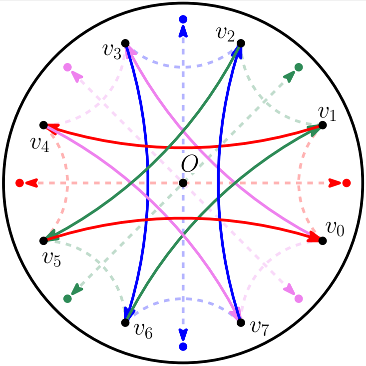
$\mathcal{G}$ contains only translations (and $\mathbb{1}$)

Bolza surface

$$\mathcal{M} = \mathbb{H}^2/\mathcal{G}$$

with projection map $\pi_{\mathcal{M}} : \mathbb{H}^2 \rightarrow \mathcal{M}$

Bolza surface



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Bolza surface

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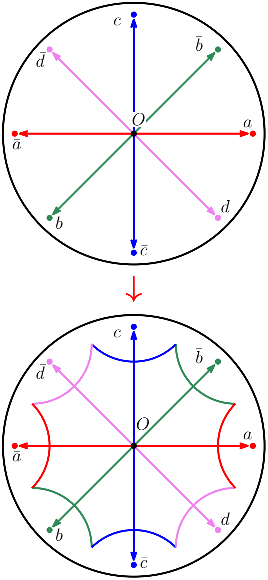
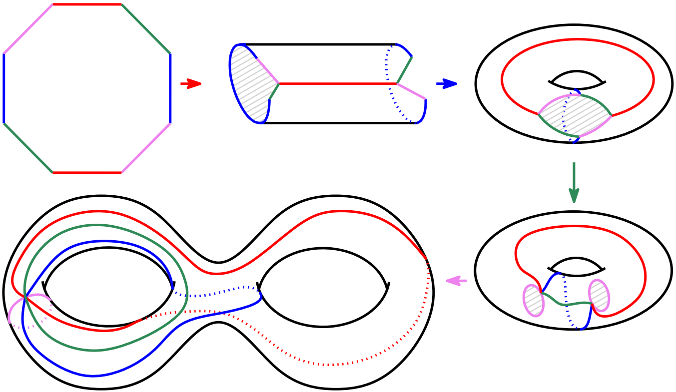
$$\mathcal{A} = \left[a, \overline{b}, c, \overline{d}, \overline{a}, b, \overline{c}, d\right] = \left[g_0, g_1, \ldots, g_7\right]$$

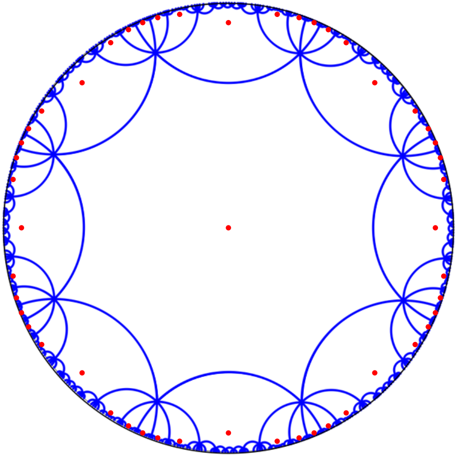
$$g_k = \begin{bmatrix} \alpha & \beta_k \\ \overline{\beta}_k & \overline{\alpha} \end{bmatrix}, \quad g_k(z) = \frac{\alpha z + \beta_k}{\overline{\beta}_k z + \overline{\alpha}}, \quad \alpha = 1 + \sqrt{2}, \quad \beta_k = e^{ik\pi/4}\sqrt{2\alpha}$$

Bolza surface



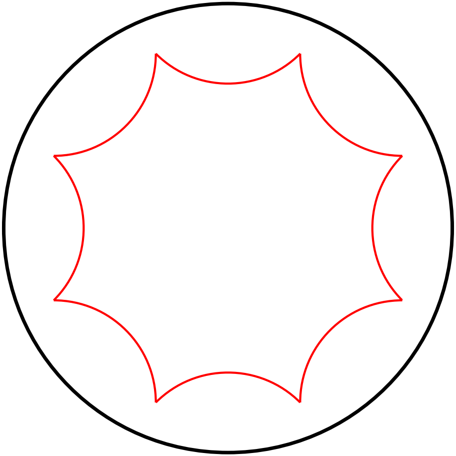


Hyperbolic octagon



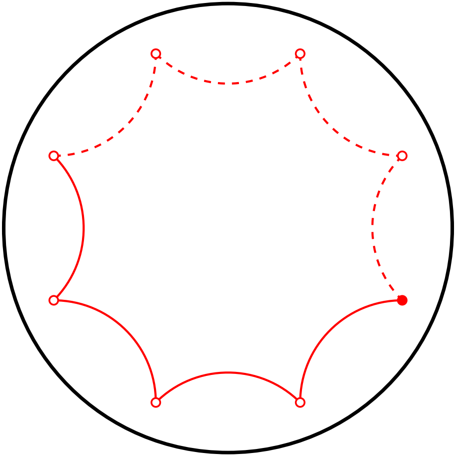
Voronoi diagram of $\mathcal{G}O$

Hyperbolic octagon



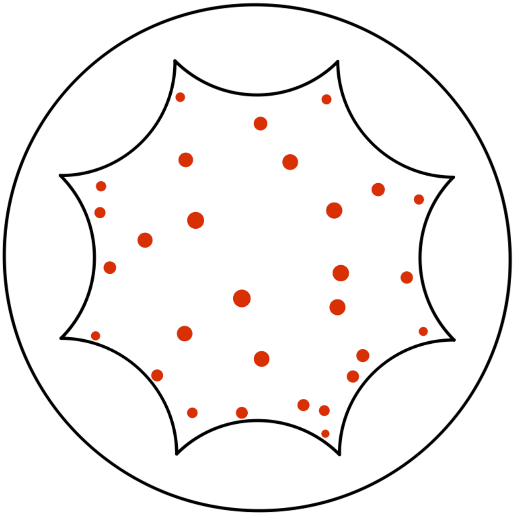
Fundamental domain $\mathcal{D}_O$ = Dirichlet region of O

Hyperbolic octagon



"Original" domain $\mathcal{D}$: contains exactly one point of each orbit

How do we triangulate the Bolza surface?



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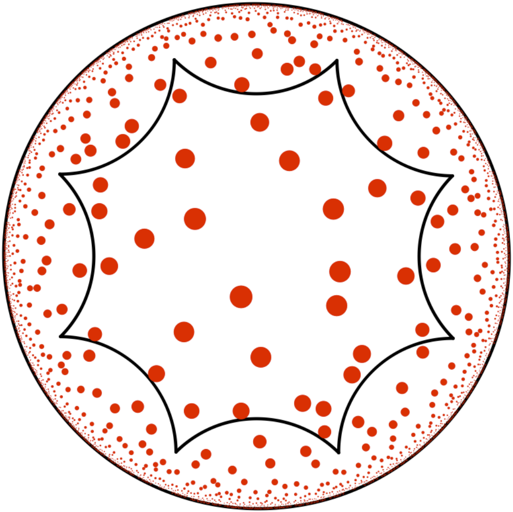

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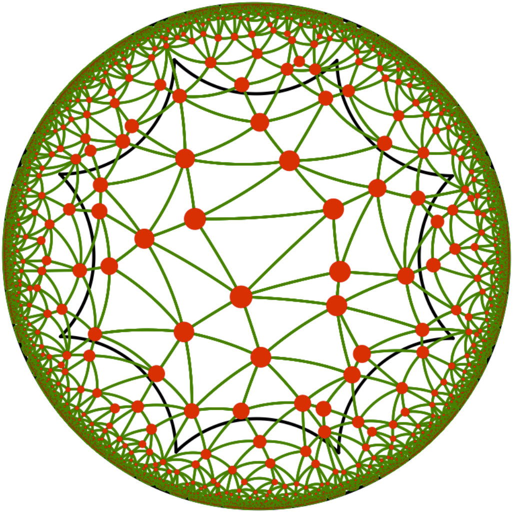

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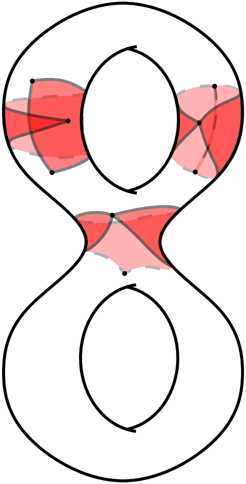

$$\pi_{\mathcal{M}}(\,DT_{\mathbb{H}}\,(\mathcal{G}S)\,)$$

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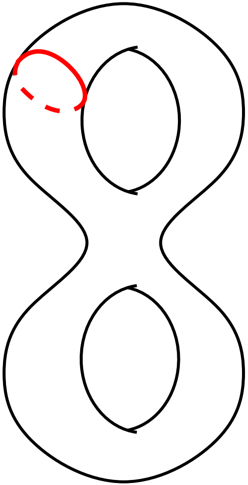

Systole $\text{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

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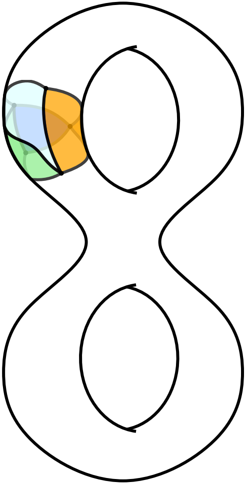


Systole $\text{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

S set of points in $\mathbb{H}^2$

$\delta_S =$ diameter of largest disks in $\mathbb{H}^2$ not containing any point of $\mathcal{G}S$

$\delta_S < \frac{1}{2}\,\text{sys}(\mathcal{M})$ [BTV16]

$\Longrightarrow \pi_{\mathcal{M}}(\,DT_{\mathbb{H}}(\mathcal{G}S)\,) = DT_{\mathcal{M}}(S)$ is a simplicial complex

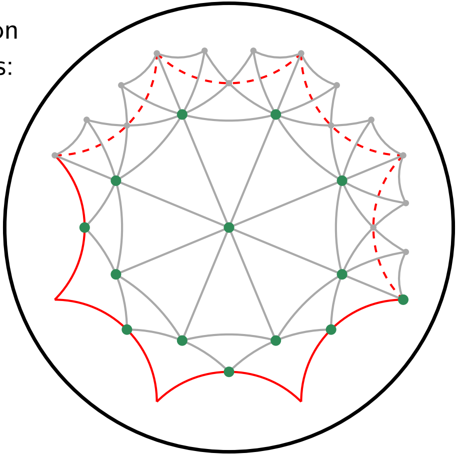
$\Longrightarrow$ The usual incremental algorithm can be used [Bowyer]

Algorithm [BTV16]

To construct the Delaunay triangulation of a point set S, we use dummy points:

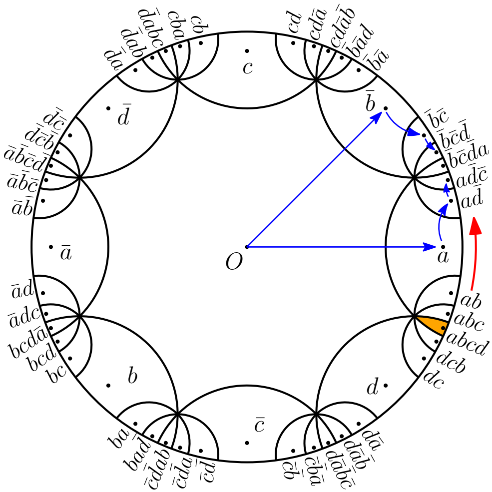
1. initialize with dummy points
2. insert points in S
3. remove dummy points

→ result may contain dummy points!



Outline

Notation



$g(O),\ g \in \mathcal{G}$, denoted as g

$\mathcal{D}_g = g(\mathcal{D}_O),\ g \in \mathcal{G}$

$\mathcal{N} = \{g \in \mathcal{G} \mid \mathcal{D}_g \cap \mathcal{D}_O \neq \emptyset\}$

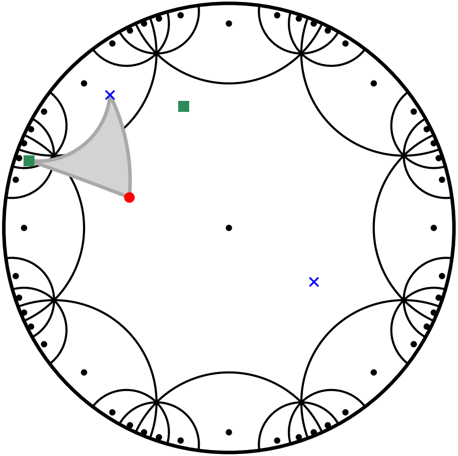
$$\mathcal{D}_{\mathcal{N}} = \bigcup_{g \in \mathcal{N}} \mathcal{D}_g$$

Property of $DT_{\mathbb{H}}(\mathcal{G}S)$

$S \subset \mathcal{D}$ input point set
s.t. criterion $\delta_S < \frac{1}{2}\,\mathrm{sys}(\mathcal{M})$ holds

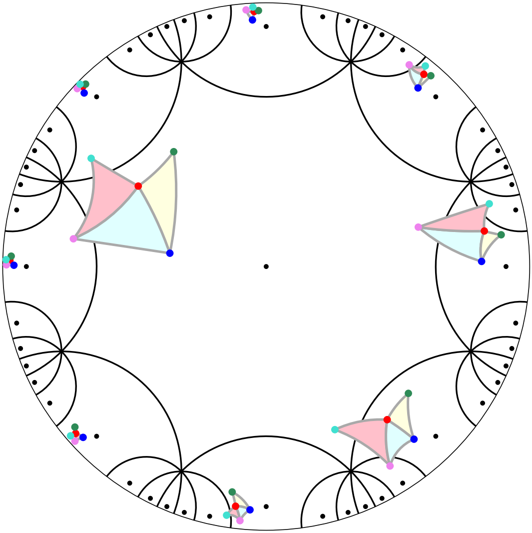
σ face of $DT_{\mathbb{H}}(\mathcal{G}S)$ with at least one vertex in $\mathcal{D}$

$\longrightarrow$ σ is contained in $\mathcal{D}_{\mathcal{N}}$



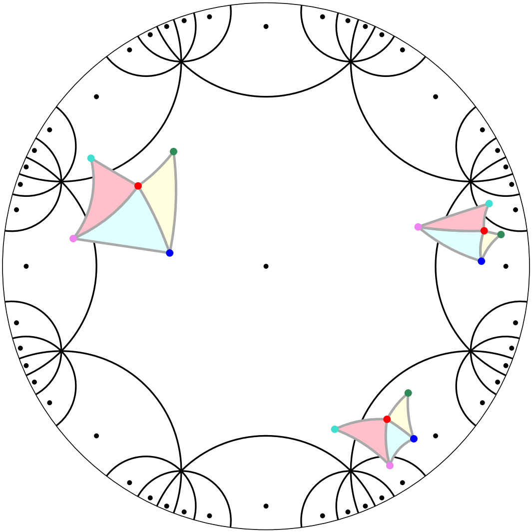
Canonical representative of a face

Each face of $DT_{\mathcal{M}}(S)$ has infinitely many pre-images in $DT_{\mathbb{H}}(\mathcal{G}S)$



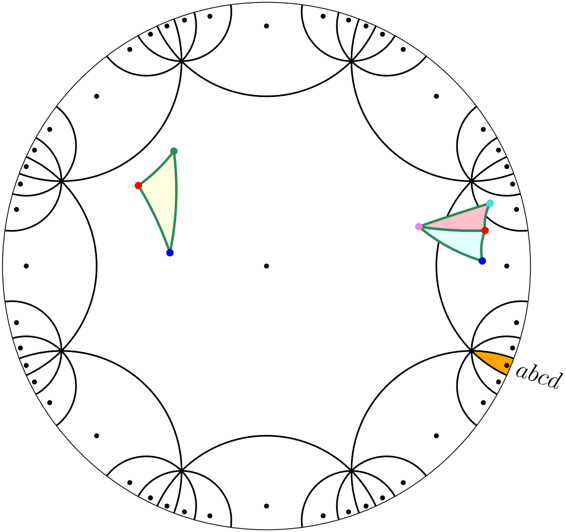
Canonical representative of a face

at least one pre-image with at least one vertex in $\mathcal{D}$

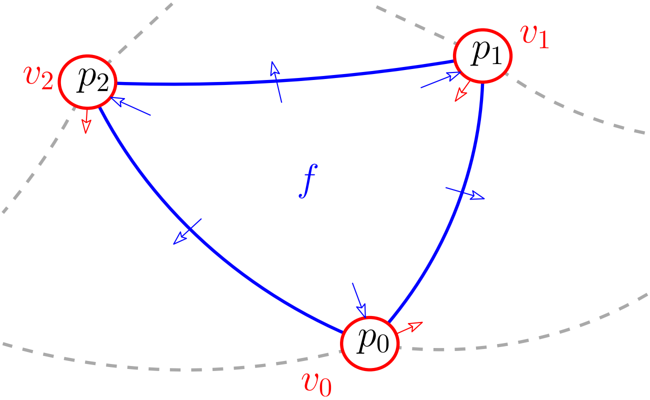


Canonical representative of a face

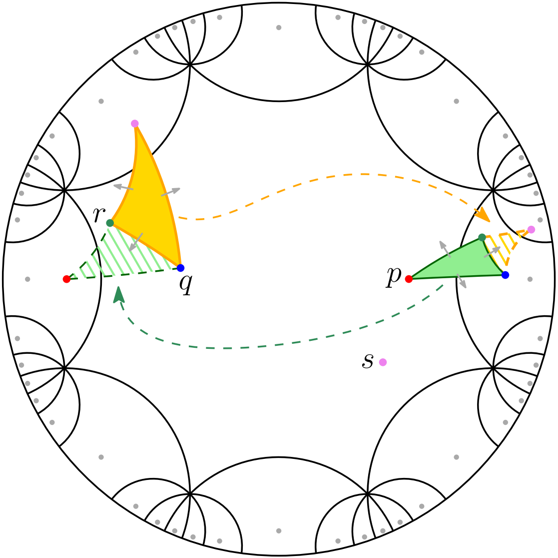
choose the pre-image "closest" to the first Dirichlet neighbor



CGAL Triangulations



Face of $DT_{\mathcal{M}}(S)$



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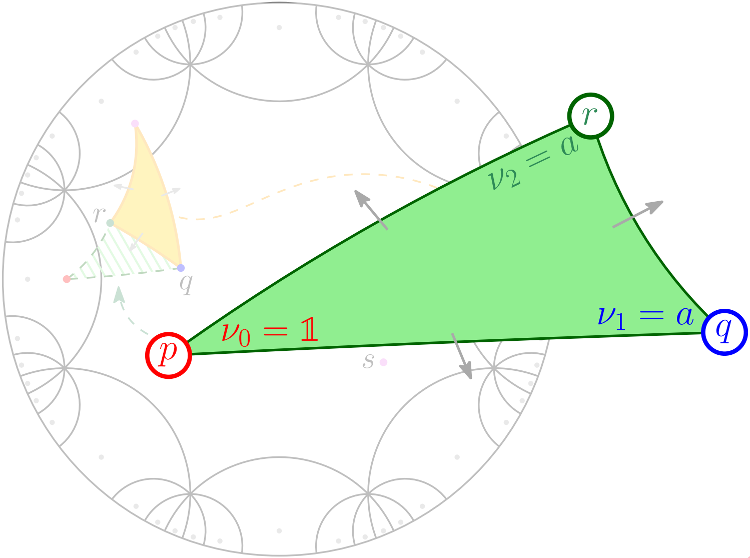
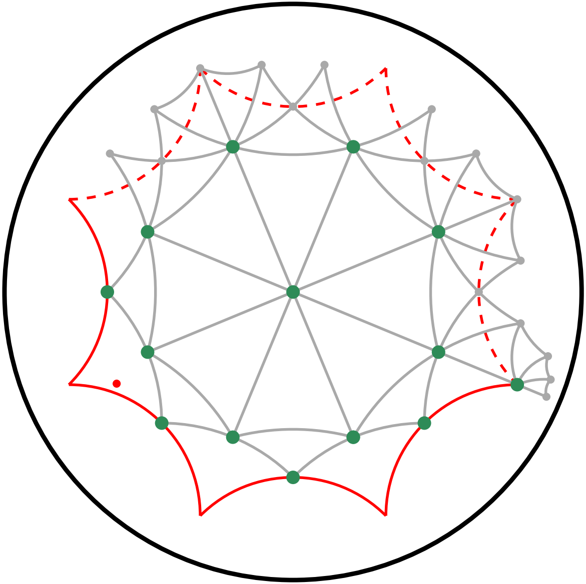


Outline

Point Location



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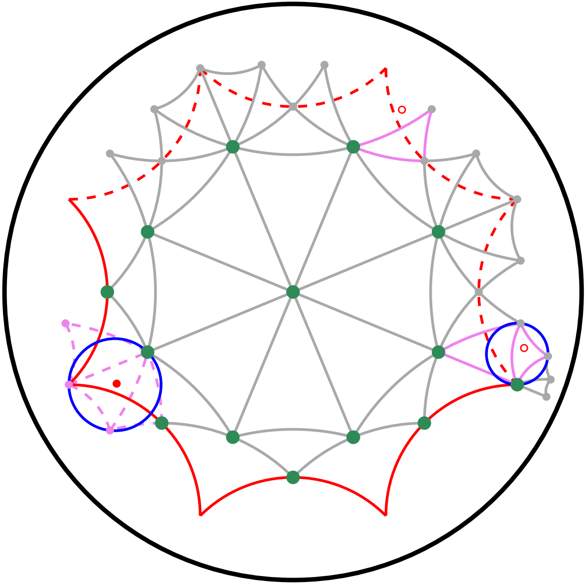

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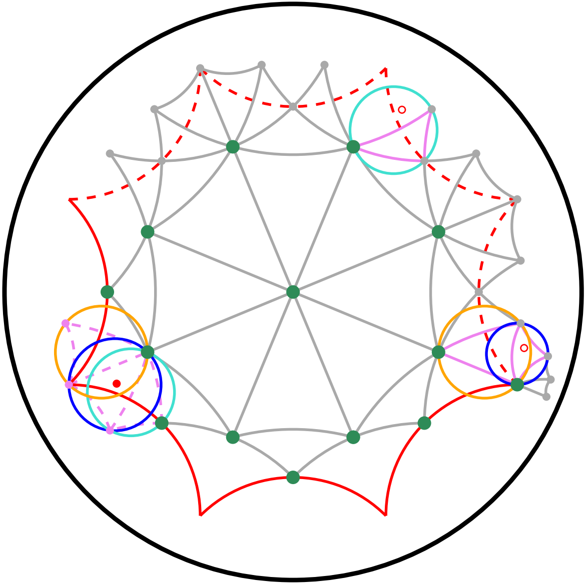

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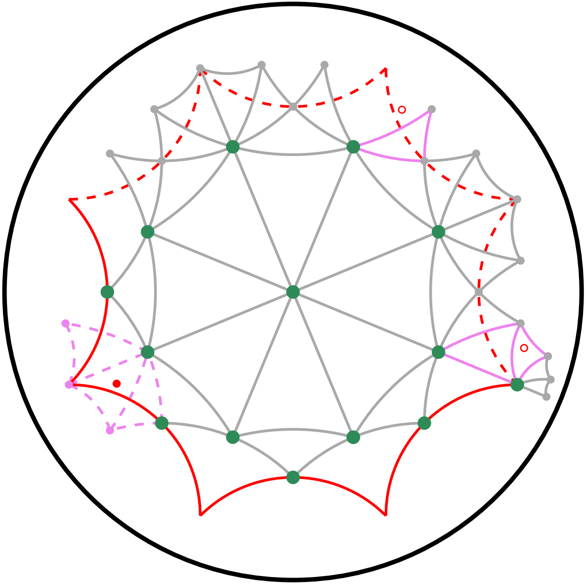
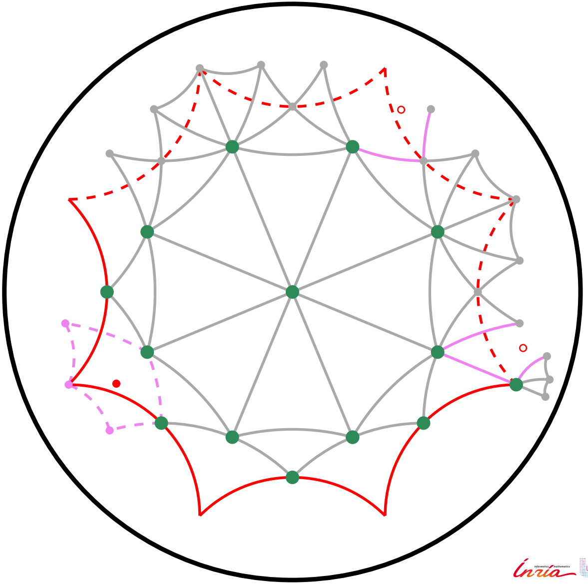


Point Insertion

“hole” = topological disk



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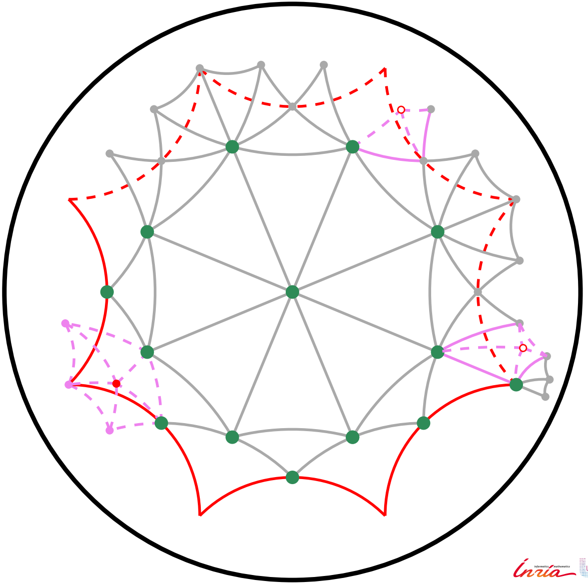
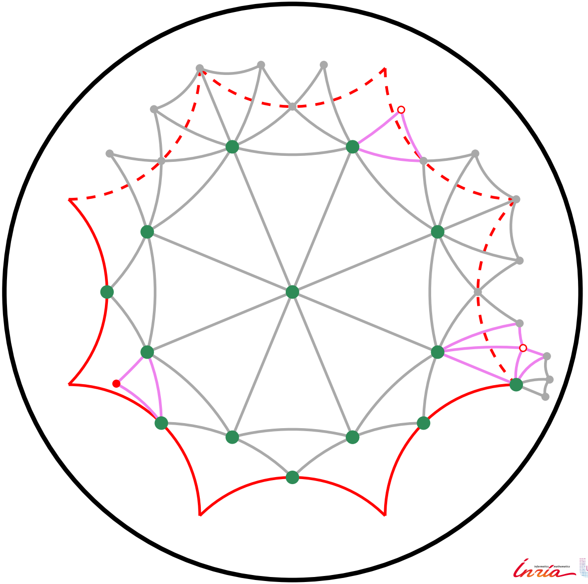


Point Insertion

Computations
on translations

Dehn's algorithm
(slightly modified)



Predicates

Suppose that the points in S are rational.

Input of the predicates can be images of these points under $\nu \in \mathcal{N}$.

$$g_k(z) = \frac{\alpha z + e^{ik\pi/4}\sqrt{2\alpha}}{e^{-ik\pi/4}\sqrt{2\alpha} z + \alpha}, \quad \alpha = 1 + \sqrt{2}, \quad k = 0, 1, ..., 7$$

- the *Orientation* predicate has algebraic degree at most 20
- the *InCircle* predicate has algebraic degree at most 72

Point coordinates represented with `CORE::Expr`
$\longrightarrow$ (filtered) exact evaluation of predicates

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Experiments

Fully dynamic implementation

→ Demo

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Fully dynamic implementation
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1 million random points

- Non-periodic triangulation
  - CGAL Euclidean DT (double) ∼ 1 sec.
  - CGAL Euclidean DT (`CORE::Expr`) ∼ 13 sec.
  - Hyperbolic DT (CK) ∼ 96 sec.
  - Hyperbolic DT (`CORE::Expr`) ∼ 265 sec.
- Periodic triangulation
  - Hyperbolic periodic DT (`CORE::Expr`) ∼ 69 sec.

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Fully dynamic implementation
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Predicates

- 0.76% calls to predicates involving translations in $\mathcal{N}$
- responsible for 36% of total time spent in predicates

Dummy points can be removed after insertion of 17–72 random points.

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Future work

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    `cgal-public-dev/Periodic_4_hyperbolic_triangulation_2-IIordanov`
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  - Submission for integration to CGAL imminent

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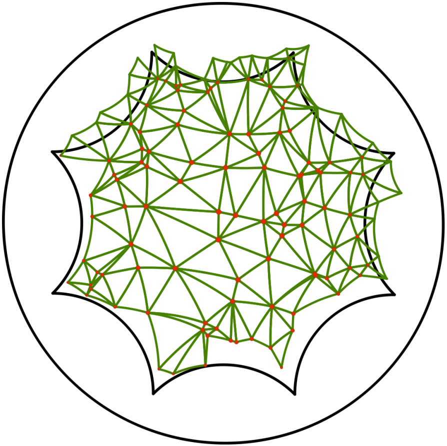
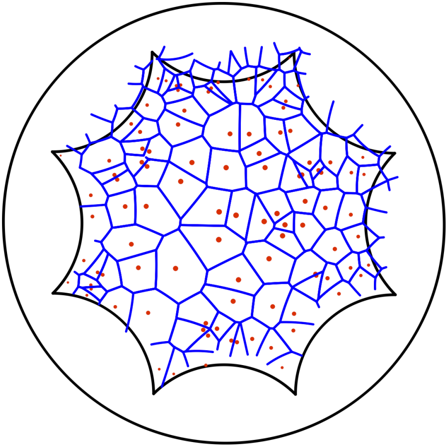
- Interface with `Mesh_2` → periodic hyperbolic mesh
- Periodic hyperbolic triangulations of regular surfaces of higher genus
  - research in progress
  - prototype code in private repository
    `INRIA/Periodic_2g_hyperbolic_triangulation_2-IIordanov`

Thank You!





Source code and Maple sheets available online:

https://members.loria.fr/Monique.Teillaud/DT_Bolza_SoCG17/