2D hyperbolic and periodic hyperbolic triangulations

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CGAL Developer Meeting, March 2018
Nancy, France
Outline

1 | Introduction
2 | Triangulations in the hyperbolic plane
3 | Periodic triangulations in the hyperbolic plane
4 | Data Structure
5 | Incremental Insertion
6 | Results
7 | Future work
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Motivation

Applications

[Sausset, Tarjus, Viot]

[Chossat, Faye, Faugeras]

[Balazs, Voros]
State of the art

Hyperbolic Delaunay simplicial complexes

- Algorithm
  [Bogdanov, Devillers, Teillaud, JoCG’14]
- Software
  [Bogdanov, Teillaud, cgal-public-dev]

Closed Euclidean manifolds

- Algorithms
  2D [Mazón, Recio], 3D [Dolbilin, Huson], dD [Caroli, Teillaud, DCG’16]
- Software (square/cubic flat torus)
  2D → Periodic_2_triangulation_2
  3D → Periodic_3_triangulation_3

Closed hyperbolic manifolds

- Algorithms
  2D, genus 2 [Bogdanov, Teillaud, Vegter, SoCG’16]
- Software (Bolza surface)
  [I., Teillaud, SoCG’17]
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Poincaré model of the hyperbolic plane $\mathbb{H}^2$
Hyperbolic Delaunay triangulations

- Hyperbolic circles $\equiv$ Euclidean circles $\Rightarrow$ reuse predicates
- If circumcircle intersects Poincaré disk, simplex is not hyperbolic! $\rightarrow$ filter out non-hyperbolic faces and edges
- Only circumcenters (Voronoi points) have non-rational coordinates
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  - Traits based on Circular_kernel_2
  - Traits with a Kernel template, to use with CORE::Expr
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  $\rightarrow$ now in Periodic_4_hyperbolic_triangulation_2-IIordanov
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- Demo
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Periodic triangulations in the Euclidean plane
Motivation

Periodic triangulations in the hyperbolic plane
Hyperbolic translations

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Hyperbolic translations

non-commutative!

I. Iordanov & M. Teillaud
Bolza surface

Fuchsian group $\mathcal{G}$ with finite presentation

$$\mathcal{G} = \langle a, b, c, d \mid abcd\bar{a}\bar{b}\bar{c}\bar{d} \rangle$$

$\mathcal{G}$ contains only translations (and $1$)

Bolza surface

$$\mathcal{M} = \mathbb{H}^2 / \mathcal{G}$$

with projection map $\pi_\mathcal{M} : \mathbb{H}^2 \to \mathcal{M}$
Bolza surface

Fuchsian group \( \mathcal{G} \) with finite presentation

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Bolza surface

\[ \mathcal{M} = \mathbb{H}^2 / \mathcal{G} \]

with projection map \( \pi_M : \mathbb{H}^2 \to \mathcal{M} \)

\[ \mathcal{A} = \begin{bmatrix} a, b, c, \bar{d}, \bar{a}, b, \bar{c}, d \end{bmatrix} = \begin{bmatrix} g_0, g_1, \ldots, g_7 \end{bmatrix} \]

\[ g_k = \begin{bmatrix} \alpha & \beta_k \\ \bar{\beta}_k & \bar{\alpha} \end{bmatrix}, \quad g_k(z) = \frac{\alpha z + \beta_k}{\beta_k z + \bar{\alpha}}, \quad \alpha = 1 + \sqrt{2}, \quad \beta_k = e^{ik\pi/4}\sqrt{2\alpha} \]
Bolza surface

Periodic triangulations in the hyperbolic plane

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Hyperbolic octagon

Voronoi diagram of $GO$
Fundamental domain $\mathcal{D}_O = \text{Dirichlet region of } O$
Hyperbolic octagon

“Original” domain $\mathcal{D}$: contains exactly one point of each orbit
How do we triangulate the Bolza surface?
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Periodic triangulations in the hyperbolic plane

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\[ \pi_M(DT_H(GS)) \]
How do we triangulate the Bolza surface?

Systole \( \text{sys}(\mathcal{M}) = \) minimum length of a non-contractible loop on \( \mathcal{M} \)

\[
\pi_\mathcal{M}(DT_H(GS))
\]
How do we triangulate the Bolza surface?

Systole $\text{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

$S$ set of points in $\mathbb{H}^2$

$\delta_S =$ diameter of largest disks in $\mathbb{H}^2$ not containing any point of $\mathcal{G}S$

$\delta_S < \frac{1}{2} \text{sys}(\mathcal{M})$ \hspace{1cm} [BTV16]

$\Rightarrow \pi_{\mathcal{M}}(DT_{\mathbb{H}}(\mathcal{G}S)) = DT_{\mathcal{M}}(S)$

is a simplicial complex

$\Rightarrow$ The usual incremental algorithm can be used \hspace{1cm} [Bowyer]
Algorithm [BTV16]

To construct the Delaunay triangulation of a point set $S$, we use dummy points:

1. initialize with dummy points
2. insert points in $S$
3. remove dummy points

→ result may contain dummy points!
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Notation

\( g(O), \ g \in \mathcal{G}, \) denoted as \( g \)

\( \mathcal{D}_g = g(\mathcal{D}_O), \ g \in \mathcal{G} \)

\( \mathcal{N} = \{ g \in \mathcal{G} \mid \mathcal{D}_g \cap \mathcal{D}_O \neq \emptyset \} \)

\[ \mathcal{D}_\mathcal{N} = \bigcup_{g \in \mathcal{N}} \mathcal{D}_g \]
Property of $\mathcal{DT}_\mathcal{H}(\mathcal{GS})$

$S \subset \mathcal{D}$ input point set
s.t. criterion $\delta_S < \frac{1}{2} \text{sys}(\mathcal{M})$ holds

$\sigma$ face of $\mathcal{DT}_\mathcal{H}(\mathcal{GS})$ with at least one vertex in $\mathcal{D}$

$\rightarrow \sigma$ is contained in $\mathcal{D}_\mathcal{N}$
Canonical representative of a face

Each face of $DT_M(S)$ has infinitely many pre-images in $DT_H(GS)$. 
Canonical representative of a face

at least one pre-image with at least one vertex in $D$
Canonical representative of a face

choose the pre-image “closest” to the first Dirichlet neighbor
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2D hyperbolic (periodic) triangulations

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Face of $DT_M(S)$
Face of $DT_M(S)$
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Point Location
Point Location
Incremental Insertion

Point Location

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2D hyperbolic (periodic) triangulations
"hole" = topological disk
“hole” = topological disk
Point Insertion

Computations on translations

Dehn’s algorithm (slightly modified)
Suppose that the points in $S$ are rational.

Input of the predicates can be images of these points under $\nu \in \mathcal{N}$.

$$g_k(z) = \frac{\alpha z + e^{ik\pi/4}\sqrt{2\alpha}}{e^{-ik\pi/4}\sqrt{2\alpha}z + \alpha}, \quad \alpha = 1 + \sqrt{2}, \quad k = 0, 1, \ldots, 7$$

- the Orientation predicate has algebraic degree at most 20
- the InCircle predicate has algebraic degree at most 72

Point coordinates represented with `CORE::Expr` → (filtered) exact evaluation of predicates
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Experiments

Fully dynamic implementation

→ Demo
Experiments

Fully dynamic implementation

→ Demo

1 million random points

- Non-periodic triangulation
  - Euclidean DT (double) \(\sim 1\) sec.
  - Euclidean DT (\texttt{CORE::Expr}) \(\sim 13\) sec.
  - Hyperbolic DT (CK) \(\sim 96\) sec.
  - Hyperbolic DT (\texttt{CORE::Expr}) \(\sim 265\) sec.

- Periodic triangulation
  - Hyperbolic periodic DT (\texttt{CORE::Expr}) \(\sim 69\) sec.
Experiments

Fully dynamic implementation

→ Demo

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Predicates

- 0.76\% calls to predicates involving translations in \(N\)
- responsible for 36\% of total time spent in predicates

Dummy points can be removed after insertion of 17–72 random points.
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Future work

- Hyperbolic and periodic hyperbolic triangulations
  - Code is public: hosted on a common branch
    cgal-public-dev/Periodic_4_hyperbolic_triangulation_2-IIordanov
  - Documentation writing is underway
  - Submission for integration to CGAL imminent
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- Interface with Mesh_2 $\rightarrow$ periodic hyperbolic mesh
- Periodic hyperbolic triangulations of regular surfaces of higher genus
  - research in progress
  - prototype code in private repository
    INRIA/Periodic_2g_hyperbolic_triangulation_2-IIordanov
Thank You!

Source code and Maple sheets available online:
https://members.loria.fr/Monique.Teillaud/DT_Bolza_SoCG17/